Fall 2008: MA Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1: Prove that the dual space of c_o is ℓ^1, where

$$c_o = \{x = (x_n) \text{such that} \lim x_n = 0\}.$$

Problem 2: Let $\{f_n\}$ be a sequence of differentiable functions on a finite interval $[a, b]$ such that the functions themselves and their derivatives are uniformly bounded on $[a, b]$. Prove that $\{f_n\}$ has a uniformly converging subsequence.

Problem 3: Let $f \in L^1(R)$ and V_f be the closed subspace generated by the translates of f: $\{f(\cdot - y) | \forall y \in R\}$. Suppose $\hat{f}(\xi_0) = 0$ for some ξ_0. Show that $\hat{h}(\xi_0) = 0$ for all $h \in V_f$. Show that if $V_f = L^1(R)$, then $\hat{f}$ never vanishes.

Problem 4: (a) State the Stone-Weierstrass theorem for a compact Hausdorff space X.

(b) Prove that the algebra generated by functions of the form $f(x, y) = g(x)h(y)$ where $g, h \in C(X)$ is dense in $C(X \times X)$.

Problem 5: For $r > 0$, define the dilation $d_r f : \mathbb{R} \longrightarrow \mathbb{R}$ of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $d_r f(x) = f(rx)$, and the dilation $d_r T$ of a distribution $T \in \mathcal{D}'(\mathbb{R})$ by

$$\langle d_r T, \phi \rangle = \frac{1}{r} \langle T, d_{1/r} \phi \rangle \qquad \text{for all test functions } \phi \in \mathcal{D}(\mathbb{R}).$$

(a) Show that the dilation of a regular distribution T_f, given by

$$\langle T_f, \phi \rangle = \int f(x) \phi(x) \, dx,$$

agrees with the dilation of the corresponding function f.

(b) A distribution is homogeneous of degree n if $d_r T = r^n T$. Show that the δ-distribution is homogeneous of degree -1.

(c) If T is a homogeneous distribution of degree n, prove that the derivative T' is a homogeneous distribution of degree $n-1$.

Problem 6: Let $\ell^2(\mathbb{N})$ be the space of square-summable, real sequences $x = (x_1, x_2, x_3, \dots)$ with norm

$$\|x\| = \left(\sum_{n=1}^{\infty} x_n^2\right)^{1/2}.$$

Define $F : \ell^2(\mathbb{N}) \to \mathbb{R}$ by

$$F(x) = \sum_{n=1}^{\infty} \left\{\frac{1}{n} x_n^2 - x_n^4\right\}$$

(a) Prove that F is differentiable at $x = 0$, with derivative $F'(0) : \ell^2(\mathbb{N}) \to \mathbb{R}$ equal to zero.

(b) Show that the second derivative of F at $x = 0$,

$$F''(0) : \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) \to \mathbb{R},$$

is positive-definite, meaning that

$$F''(0)(h, h) > 0$$

for every nonzero $h \in \ell^2(\mathbb{N})$.

(c) Show that F does not attain a local minimum at $x = 0$.