Fall 2001 Mathematics Graduate Program Masters Exam

Instructions: Explain your answers clearly. Unclear answers will not receive credit. State results and theorems that you are using.

1. Algebra

Problem 1. a. Prove Lagrange’s theorem: If $G$ is a finite group and $H$ is a subgroup then $|H|$ divides $|G|$.  
Prove or disprove: If $n$ divides $|G|$ then there is a subgroup of $G$ of order $n$.

Problem 2. a. Give an example of a group and a subgroup which is not normal.
b. Show that a group $G$ of order 33 has a subgroup $H$ of order 11.
c. Show that this subgroup $H$ is normal.

Problem 3. a. Give an example of a noncommutative ring.
b. Let $\mathbb{Z}[x]$ denote the ring of polynomials with integer coefficients. Prove or disprove: $\mathbb{Z}[x]$ is a Principle Ideal Domain.

Problem 4. a. Give three examples of Field extensions of the rationals $\mathbb{Q}$.
b. Let $\mathbb{Q}(a)$ denote the field extension of the rationals obtained by adjoining $a$. Show that the field $\mathbb{Q}(\sqrt{2})$ is not isomorphic to $\mathbb{Q}(\sqrt{3})$.

Problem 5. a. Give an example of a finite field of order 3 and a finite field of order 9.
b. Give an example of an infinite field of characteristic 3.
c. Let $F$ be a finite field. Show that the order of $F$ is equal to $p^n$ for some prime number $p$ and positive integer $n$.

2. Analysis

Problem 6. Let $f : R \to R$ be a differentiable mapping with 
$$\lim_{x \to \infty} f(x) = 0.$$  
a. Show that there exists a sequence $x_n \to \infty$ with 
$$\lim_{n \to \infty} f'(x_n) = 0.$$

b. Show that it is not necessarily true that $f'(x)$ is bounded.

Problem 7. Let $f : [0, 1] \to [0, 1]$ be a continuous function. Show that $f(x) = x$ for some $x$.
Is the same true for a continuous function $f : (0, 1) \to (0, 1)$ on the open unit interval? Prove or give a counterexample.

Problem 8. Suppose $\lim_{n \to \infty} p_n = p_0$. Show that the set $E = \{p_0, p_1, p_2, \ldots\}$ is compact.

Problem 9. Prove that $C[0,1]$, the space of continuous functions on $[0,1]$, is not complete in the $L^1$ metric: 
$$\rho(f, g) = \int |f(x) - g(x)| \, dx$$
Problem 10. Consider the map \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) given by

\[
T_1(x_1, x_2, x_3) = y_1 = x_1 \cos x_2 \\
T_2(x_1, x_2, x_3) = y_2 = x_1 \sin x_2 \\
T_3(x_1, x_2, x_3) = y_3 = x_3
\]

a. Compute the Jacobian matrix of \( T \).
b. For which values of \( x = (x_1, x_2, x_3) \) is the map locally invertible (i.e. there exists a neighborhood \( U \) of \( x \) and a neighborhood \( V \) of \( y = T(x) \) such that \( T : U \to V \) is 1-1 and onto with inverse map \( T^{-1} : V \to U \)).
c. Compute the Jacobian matrix of \( T^{-1} \) at \( f(x) \) where it exists.

3. Linear Algebra and Other Areas

Problem 11. a. Give an example of a real \( n \times n \) matrix none of whose eigenvalues are real numbers.
b. Show that there is no such example which is \( 3 \times 3 \).
c. Show that every eigenvalue of a symmetric real matrix is real.

Problem 12. a. Give three examples of linear mappings \( L : \mathbb{R}^3 \to \mathbb{R}^3 \) satisfying \( L^2 = L \).
Let \( P : \mathbb{R}^n \to \mathbb{R}^n \) be a linear mapping satisfying \( P^2 = P \).
a. Show that every vector in \( \mathbb{R}^n \) can be written as a sum of two vectors, one in the kernel of \( P \) and one in the image of \( P \).
b. If \( P \) is symmetric (self-adjoint) then show that the image of \( P \) and the kernel of \( P \) are orthogonal subspaces of \( \mathbb{R}^n \).