PRELIMINARY EXAM IN ANALYSIS
FALL 2017

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

1. Prove that every metric subspace of a separable metric space is separable.

2. Let X be a Banach space with dual space X^* and let $A \subseteq X$ be a linear subspace. Define the annihilator $A^\perp \subseteq X^*$ of A by

$$A^\perp = \{f \in X^* : f(x) = 0 \text{ for all } x \in A\}.$$

Prove that A is dense in X if and only if $A^\perp = \{0\}$.

3. Prove or disprove the following statement: If $f \in C^\infty([0,1])$ is a smooth function, then there is a sequence (p_n) of polynomials on $[0,1]$ such that $p_n^{(k)} \to f^{(k)}$ uniformly on $[0,1]$ as $n \to \infty$ for every integer $k \geq 0$. Here $f^{(k)}$ denotes the kth derivative of f.

4. Let $[a,b] \subseteq \mathbb{R}$ be a closed interval and let

$$\|f\|_\infty = \sup_{x \in [a,b]} |f(x)| \qquad \|f\|_2 = \sqrt{\int_a^b |f(x)|^2 dx}$$

denote the L^∞ and L^2 norms. If $f \in C^1([a,b])$, prove that

$$\|f\|_\infty^2 \leq \frac{\|f\|_2^2}{b-a} + 2\|f\|_2 \, \|f'\|_2.$$

5. Let A be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$. Prove that

$$\exp(iA) = \sum_{n=0}^{\infty} \frac{1}{n!}(iA)^n$$

is also a well-defined bounded operator on $\mathcal{H}$ and is unitary.

6. Consider a linear functional $\phi(f) = f(1/2)$ defined on the space of polynomials on $[0,1]$. Does ϕ extend to a bounded linear functional on $L^2([0,1])$? Prove or disprove.