Instructions:
1. All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.
2. Use separate sheets for the solution of each problem.

Problem 1.
Let $G_1, G_2$ be finite index subgroups of a group $G$. Show that the intersection $G_1 \cap G_2$ also has finite index in $G$.

Problem 2.
Let $G$ be a finite group and $N \subseteq G$ be a subgroup of index $p$, where $p$ is the smallest prime dividing $|G|$. Prove $N$ is a normal subgroup of $G$ ($N \triangleleft G$).

Problem 3.
Does the additive group $\mathbb{Q}$ admit an epimorphism to a nontrivial finite group? Justify your answer.

Problem 4.
List all ideals of $\mathbb{F}_p[x]/(x^2 + x - 6)$ when
(a) $p = 7$
(b) $p = 5$.

Problem 5.
Let $\rho$ be a representation of a finite group $G$ on a vector space $V$ and let $v \in V$.
(a) Show that averaging $\rho_g(v)$ over $G$ gives a vector $\overline{v} \in V$ which is fixed by $G$.
(b) What can you say about this vector when $\rho$ is an irreducible representation?

Problem 6.
If $R$ is a commutative ring with identity, and $S$ a multiplicative subset, then every ideal $J$ of $S^{-1}R$ is of the form $S^{-1}I$ for some ideal $I$ of $R$. Is $I$ uniquely determined by $J$? Why or why not?