MAT 21C Final Exam

Name: ________________________________
Student ID #: __________________________
Time and Day of Your Discussion Section#: __________________
Name of Left Neighbor: _______________________
Name of Right Neighbor: _______________________

If you are next to the aisle or wall, then please write “aisle” or “wall” appropriately as your left or right neighbor.

- Read each problem carefully.
- Write every step of your reasoning clearly.
- Usually, a better strategy is to solve the easiest problem first.
- This is a closed-book exam. You may not use the textbook, crib sheets, notes, or any other outside material. Do not bring your own scratch paper. Do not bring blue books.
- No calculators/laptop computers/cell phones are allowed for the exam. The exam is to test your basic understanding of the material.
- Everyone works on their own exams. Any suspicions of collaboration, copying, or otherwise violating the Student Code of Conduct will be forwarded to the Student Judicial Board.

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<th>Problem #</th>
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Problem 1 (15 pts) Does the following series converge or diverge? Give reasons for your answer.

(a) (7 pts)
\[ \sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}. \]

(b) (8 pts)
\[ \sum_{n=1}^{\infty} \frac{n!}{n^n}. \]

Hint: You may want to use the fact:
\[ \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^{n} = e \approx 2.718. \]
Problem 2 (10 pts) Find the radius and the interval of convergence of the series

\[ \sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}. \]
Problem 3 (10 pts) Consider the function \( f(x) = \sqrt{1 + x} \).

(a) (5 pts) Approximate \( f(x) \) by a Taylor polynomial of degree 1 centered at 0. Compute the value of \( \sqrt{2} \) using that approximation.

(b) (5 pts) How accurate is this approximation when \( 0 \leq x \leq 1 \)? You can express the approximation error either by fraction or by three decimal point number. Then, confirm that the approximate value of \( \sqrt{2} \) computed in Part (a) is within this error. Note that the precise value of \( \sqrt{2} \) up to 4 digits is 1.414.
Problem 4 (10 pts) Let $P(1, 4, 6), Q(-2, 5, -1), R(1, -1, 1)$.

(a) (5 pts) Find the area of the triangle $\triangle PQR$.
   
   Hint: Length of the cross product of two vectors is equal to the area of a parallelogram formed by those two vectors.

(b) (5 pts) Find the distance from $P$ to the line $QR$. 

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Problem 5 (10 pts) Find the limit if it exists, or show that the limit does not exist.

(a) (5 pts) 
\[ \lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^4 + y^4}. \]
Hint: Consider \((x, y) \to (0, 0)\) along the line \(y = x\) and along the parabola \(y = x^2\).

(b) (5 pts)
\[ \lim_{(x,y) \to (0,0)} \frac{x^3 + y^3}{x^2 + y^2}. \]
Hint: Consider the limit in the polar coordinates \((r, \theta)\).
Problem 6 (10 pts) Verify that the function of two variables

\[ u(x, t) = e^{-\alpha^2 k^2 t} \sin kx \]

is a solution of the heat conduction equation:

\[ u_t = \alpha^2 u_{xx}. \]
**Problem 7** (10 pts) Assuming that the equation

\[ xe^y + \sin(xy) + y - \ln 2 = 0 \]

defines \( y \) as a differentiable function of \( x \), use the Implicit Differentiation Theorem to find the value of \( \frac{dy}{dx} \) at the point \( (x, y) = (0, \ln 2) \).
Problem 8 (15 pts) Find the direction in which \( f(x, y) = \sin x + e^{xy} \)

(a) (5 pts) Increases most rapidly at the point \((0, 1)\).

(b) (5 pts) Decreases most rapidly at the point \((0, 1)\).

(c) (5 pts) Does not change (i.e., is flat) at the point \((0, 1)\).
Problem 9 (15 pts) Consider the sphere with radius $r > 0$ in 3D, $x^2 + y^2 + z^2 = r^2$.

(a) (7 pts) Find the tangent plane at the point $\left(\frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}\right)$ on this sphere.

(b) (8 pts) Show that every normal line to this sphere passes through the center of the sphere, i.e., the origin.
   Hint: Pick any point $(a, b, c)$ on this sphere, and consider the normal line at that point.

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Problem 10 (15 pts) Let \( f(x, y) = 2x^2 + y^2 \).

(a) (8 pts) Find the linearization at the point \((1, 1)\). Then use it to approximate \( f(1.1, 0.9) \). Compare the approximate value with the true value.

(b) (7 pts) Approximate \( f(2, 2) \) using the same linearization. Compare the approximate value with the true value. At which point is the linear approximation better, \((1.1, 0.9)\) or \((2, 2)\)?
Problem 11 (15 pts) Consider the following function over the closed domain $D = \{(x, y) \mid -\pi \leq x \leq \pi, -\pi \leq y \leq \pi\}$: 

$$f(x, y) = x \cos y.$$ 

(a) (7 pts) Find the local maxima, local minima, saddle points of $f$ if any.

(b) (8 pts) Find the absolute maxima and absolute minima.
Problem 12 (15 pts) Use Lagrange multipliers to find the maximum and minimum values of the function

\[ f(x, y) = e^{xy} \quad \text{subject to} \quad x^2 + y^2 = 1. \]

Note that we only consider the real values for \( x \) and \( y \), not the complex values.