

ANALYSIS PRELIM PROBLEMS  
Spring, 2015

**Instructions:**

1. All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.
2. Use separate sheets for the solution of each problem.

**Problem 1.** Let  $\mathcal{H}$  be a separable Hilbert space. A sequence  $(x_n)$  in  $\mathcal{H}$  converges in the Cesàro sense to  $x \in \mathcal{H}$  if the averages of its partial sums converge strongly to  $x$ , i.e., if

$$\bar{x}_N = \frac{1}{N} \sum_{n=1}^N x_n \rightarrow x \quad \text{as } N \rightarrow \infty.$$

- (a) Prove that if  $(x_n)$  converges strongly to  $x$  in  $\mathcal{H}$ , then  $(x_n)$  also converges in the Cesàro sense to  $x$ .
- (b) Give an example of a sequence that converges in the Cesàro sense but does not converge weakly.
- (c) Give an example of a sequence that converges weakly but does not converge in the Cesàro sense.

**Problem 2.**

Suppose  $f : [-1, 1] \rightarrow \mathbb{R}$  is an odd continuous function such that  $f(-1) = f(1)$ . Given that  $\int_{-1}^1 \sin(nx)f(x)dx = 0$  for all positive integers  $n$ , show that  $f \equiv 0$ .

**Problem 3.**

Let  $P(x) : \mathbb{R} \rightarrow \mathbb{R}$  be a polynomial of degree  $n$ . Show that there exists a constant  $C$  depending only on  $n$  such that  $|P(\xi)| \leq C \int_{-1}^1 |P(x)|^2 dx$  for all  $\xi \in (-1, 1)$ . (Remark: It may help to consider this problem from a functional analytic perspective.)

**Problem 4.**

Let  $f_n : [0, 1] \rightarrow \mathbb{R}$  be a sequence of measurable functions. Suppose

- (i)  $\int_0^1 |f_n(x)|^2 dx \leq 1$  for  $n = 1, 2, \dots$ .

(ii)  $f_n \rightarrow 0$  almost everywhere.

Show that

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = 0.$$

**Problem 5.**

Find the Fourier series of the  $2L$ -periodic extension of

$$f(x) = \begin{cases} x & \text{if } x \in [0, L] \\ 0 & \text{if } x \in (-L, 0] \end{cases}$$

Show that

$$\pi^2/8 = \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}.$$

**Problem 6.**

Let  $\{A_n\}$  be a sequence of bounded linear operators on a Hilbert space  $H$  that converges weakly to an operator  $A$ , and suppose that for each  $x \in H$  one has  $\|A_n x\| \rightarrow \|Ax\|$  as  $n \rightarrow \infty$ . Prove that  $A_n$  strongly converges to  $A$  (in particular, for unitary operators weak convergence to a unitary operator implies strong convergence).