Spring 2013: PhD Algebra Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1: Let A be a *Boolean ring*, i.e., $a^2 = a$ for all $a \in A$. Show that the ring A is commutative.

Problem 2: Let R be a commutative ring with identity $1_R \neq 0$. Let $I \subset R$ be an ideal so that R/I is a division ring. Show that I is a maximal ideal in R.

Problem 3: Suppose that n and m are natural numbers. Show that the free group of rank n is isomorphic to the free group of rank m if and only if $n = m$.

Problem 4: Let E/K be a field extension of degree $[E : K] = 2^k$ for some $k \geq 1$. Let $f \in K[X]$ be a monic polynomial of degree 3 that has a root in E. Must f have a root in K?

Problem 5: Consider the multiplicative group $\mathbb{F}_{13}^{\times}$ of the field $\mathbb{F}_{13}$. Which elements generate the group, and which elements are squares in $\mathbb{F}_{13}^{\times}$?

Problem 6: Let G be a group. Prove or disprove the following statements.
(a) If G is abelian, then every finite-dimensional irreducible complex representation of G is one-dimensional.
(b) If G is finite and every irreducible complex representation of G is one-dimensional, then G is abelian.