PRELIMINARY EXAM IN ALGEBRA
FALL, 2016

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

1. Let F_n be the free group on n generators with $n \geq 2$. Prove that the center $Z(F)$ of F is trivial.

2. Let G be a finite group that acts transitively on a set X of cardinality ≥ 2. Show that there exists an element $g \in G$ which acts on X without any fixed points. Is the same true if G is infinite?

3. Show that every linear map $A : \mathbb{R}^3 \to \mathbb{R}^3$ has both a 1-dimensional invariant subspace and a 2-dimensional invariant subspace.

4. Let $I, J \subseteq R$ be ideals in a principal ideal domain R. Prove that $I + J = R$ if and only if $IJ = I \cap J$.

5. Let F be a finite field and let L be the subfield of F generated by elements of the form x^3 for all $x \in F$. Prove that if $L \neq F$, then F has exactly 4 elements.

6. Show that the $\mathbb{R}$-modules $L = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $M = \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are not isomorphic.