Problem 1 (10 points). *Bring the following integer matrix A to its canonical form* $\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \end{bmatrix}$ *by multiplying elements of $GL_3(\mathbb{Z})$ from the left, and elements of $GL_4(\mathbb{Z})$ from the right, where the entries satisfy $0 < d_1$, and $d_1 | d_2 | d_3$.*

$$A = \begin{bmatrix} 2 & 2 & 2 & 2 \\ & 3 & 3 & 3 \\ & & 4 & 4 \end{bmatrix}.$$

Identify the structure of the $\mathbb{Z}$-module V presented by $A : \mathbb{Z}^4 \longrightarrow \mathbb{Z}^3$.

Problem 2 (10 points). *This problem concerns basic definitions.*

(1) [3 points] Let R be a commutative ring with 1. Give the definition of a **finitely generated** R-module V.
(2) [3 points] Let F be a field and $f(x) \in F[x]$ a polynomial. State the definition of a **splitting field** of $f(x)$ over F.
(3) [4 points] Let K be a finite extension field of another field F. State the definition that K is a **Galois extension** of F.

Problem 3 (10 points). *Let a and b be two positive integers that are relatively prime. Prove that as a $\mathbb{Z}$-module, $\mathbb{Z}/(a) \oplus \mathbb{Z}/(b)$ and $\mathbb{Z}/(ab)$ are isomorphic:*

$$\mathbb{Z}/(a) \oplus \mathbb{Z}/(b) \cong \mathbb{Z}/(ab),$$

where $(a) \subset \mathbb{Z}$ denotes the ideal generated by $a \in \mathbb{Z}$. You can use any method you like.

Problem 4 (10 points, 2 points each). *Let $\omega = e^{2\pi i/3}$ be a primitive cubic root of unity, and let $K = \mathbb{Q}(\sqrt{2}, \omega)$ be an algebraic extension of $\mathbb{Q}$. Solve the following problems. Give your reason for each of the problems.*

(1) Find the extension degree $[K : \mathbb{Q}]$.
(2) Prove that K is a Galois extension of $\mathbb{Q}$.
(3) Identify the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.
(4) Find all distinct subgroups of $\mathrm{Gal}(K/\mathbb{Q})$.
(5) List all intermediate fields of the extension $\mathbb{Q}(\sqrt{2}, \omega)$ over $\mathbb{Q}$.

Problem 5 (10 points, 2 points each). *Let K be a finite field of order $q = 125$. Answer each of the following questions. Give your reason as well.*

(1) What is the characteristic $p = \mathrm{ch}(K)$ of the field K?
(2) For the characteristic you have determined in the above, what is the extension degree $[K : \mathbb{F}_p]$?
(3) What is the multiplicative group $K^\times$? Identify its structure.
(4) There is a monic positive degree polynomial $f(x) \in \mathbb{F}_p[x]$ such that every element $\alpha \in K$ satisfies that $f(\alpha) = 0$. What is it?
(5) What is 3^{125} in $\mathbb{Z}/(5)$?

Problem 6 (10 points). *A $\mathbb{Z}$-module V is called* simple *if it is not the zero module $\{0\}$ and has no non-trivial $\mathbb{Z}$-submodules other than V itself. Classify all simple $\mathbb{Z}$-modules.*