ON NETWORK DESIGN UNDER UNCERTAIN ARC CAPACITIES

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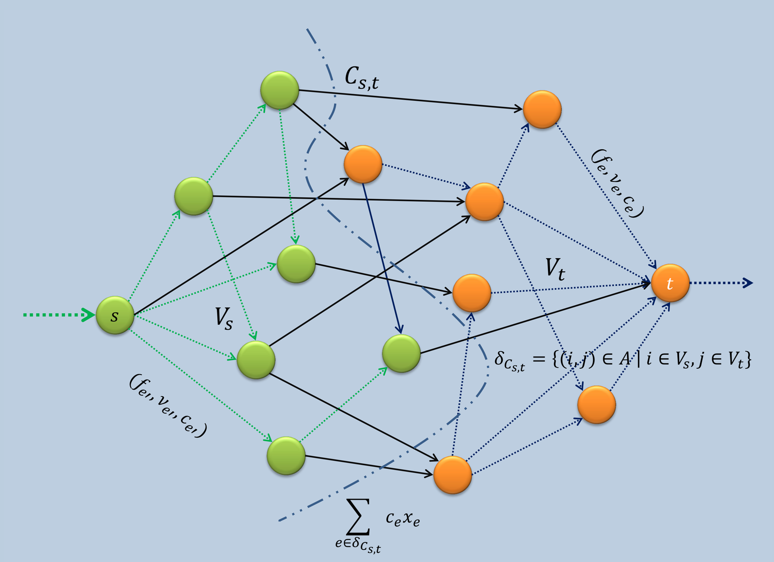
Contribution

We introduce robust models to determine minimum cost network setup(s) with particular attention to the stochastic nature of the arc capacities. An intrinsic motivation is that the structure (topology) of the network itself is not deterministic. The network setup(s) thus desired need to be not only cost efficient, but also robust with respect to potential arc capacity disruptions. Specifically we address the important question:

How to efficiently minimize the costs of network design in the framework of large-scale fixed charge network flow problems with uncertain arc capacities?

Problem Formulation



Consider a network $N(V,A)$, $|A| = m$, $|V| = n$, with uncertain (independent) arc capacities ξ_e, $e \in A$. we find optimal solution(s), whilst analyzing any difficulties during the solution procedure of,

$$\begin{aligned}
&\min \quad \mathbf{f}'\mathbf{x} && (1)\\
&s.t \quad \mathbf{N}\mathbf{y} = \mathbf{d} && (2)\\
&\qquad \mathbf{Pr}\left(\sum_{e\in\delta_{C_{s,t}}} \xi_e x_e \ge d\right) \ge 1-\epsilon \quad \forall\, C_{s,t} \in \mathcal{C} && (3)\\
&\qquad 0 \le y_e \le u_e x_e \qquad \forall\, e \in A && (4)\\
&\qquad \mathbf{x} \in \{0,1\}^m && (5)
\end{aligned}$$

where,
u_e : nominal value of ξ_e, $\forall\, e \in A$
$1-\epsilon$: desired confidence level $(0 \le \epsilon \le 0.5)$

Under *certain* conditions [*cf.* Bertsimas and Popescu, 2005; Ghaoui *et al.*, 2003; Ben-Tal and Nemirovski, 1991, 2000, 2002] constraints of type (3) (or approximations thereof) can be modeled as conic-quadratic $0-1$ constraints,

$$\mathbf{u}'\mathbf{x} - \Omega(\epsilon)\|\mathbf{\Sigma}\mathbf{x}\| \ge d \tag{6}$$

where,
σ_e : deviation statistic for $\xi_e, e \in A$,
$\mathbf{\Sigma} = \mathrm{diag}\{\sigma_1,\sigma_2,\ldots,\sigma_m\}$, $\Omega(\epsilon) > 0$.
$\Omega(\epsilon)\|\mathbf{\Sigma}\mathbf{x}\|$: used to build sufficient slack into the constraint to accommodate variability of ξ_i around u_i.

Separation of conic inequalities

Observe, for a parameterized function Ψ,

$$\Psi(\mathbf{x}, C_{s,t}) \ge 1 \;\forall\; C_{s,t} \in \mathcal{C} \iff \min_{C_{s,t}\in\mathcal{C}} \Psi(\mathbf{x}, C_{s,t}) \ge 1$$

$\Psi : [0,1]^m \mapsto \mathbb{R}$ is defined as,

$$\Psi(\mathbf{x}) := \min_{C_{s,t}\in\mathcal{C}}\left(\sum_{e\in\delta_{C_{s,t}}} u_e x_e - \Omega\sqrt{\sum_{e\in\delta_{C_{s,t}}} \sigma_e^2 x_e^2}\right) \tag{7}$$

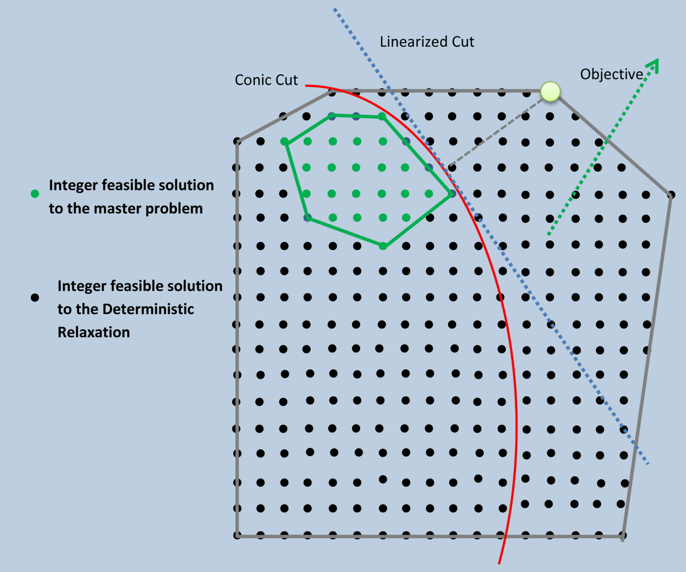
using the fact that x is binary, we can obtain an exact convex reformulation,

$$\Psi(\mathbf{x}) := \min_{C_{s,t}\in\mathcal{C}}\left(\sum_{e\in\delta_{C_{s,t}}} u_e x_e - \Omega\sqrt{\sum_{e\in\delta_{C_{s,t}}} \sigma_e^2 x_e}\right) \tag{8}$$

Linearizations

A hyperplane, separating the integer feasible set from infeasible solution p,

$$(\mathbf{x}-\bar{\mathbf{x}})'\nabla f(\bar{\mathbf{x}}) \le 0, \quad \bar{\mathbf{x}} = \mathbf{Proj}_f(\mathbf{p}). \tag{9}$$



In particular we add the linear inequality,

$$\sum_{e\in\delta_{C_{s,t}}}\left(u_e - \Omega\frac{\sigma_e^2 \bar{x}_e}{\sqrt{\bar{\mathbf{x}}'\mathbf{\Sigma}\bar{\mathbf{x}}}}\right)(x_e - \bar{x}_e) \ge 0 \tag{10}$$

Polyhedral Study

Underlying *supermodular min-knapsack polytope*,

$$X_{CQ} := \{\mathbf{x} \in \{0,1\}^m \mid f(\mathbf{x}) \ge d\} \tag{11}$$

Specifically here, $f(\mathbf{x}) = \mathbf{u}'\mathbf{x} - \Omega\|\mathbf{\Sigma}\mathbf{x}\|$ Define, difference function with respect to the set function f as,

$$\rho_i(S) := f(S \cup i) - f(S) \text{ for } S \subseteq N \setminus i$$

f supermodular $\Leftrightarrow \rho_i(S) \le \rho_i(T)\ \forall S \subseteq T \subseteq N \setminus i$ and $i \in N$.

• Pack Inequalities

A subset P of N is said to be a *pack* for X if $\delta := d - f(P) > 0$.
A pack P is maximal if

$$f(P \cup i) \ge d, \quad \forall\, i \in N \setminus P \tag{12}$$

Let, $X^P = \{\mathbf{x} \in X \mid x_i = 1, \text{ for } i \in P\}$.
If $P \subset N$ is a pack for X, then pack inequality

$$\mathbf{x}(N \setminus P) \ge 1 \tag{13}$$

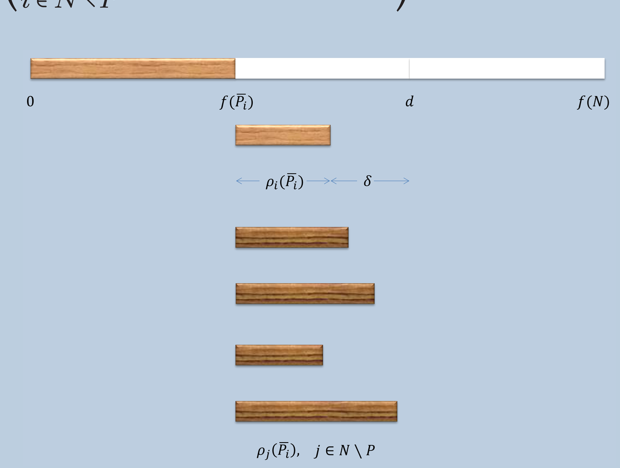
is valid for X. Moreover, (13) defines a facet of $\mathrm{conv}(X^P)$ if and only if P is a maximal pack.

• Extended Pack Inequalities

Let $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_{|P|})$ be a permutation of the elements of P. Define, $\bar{P}_i := P \setminus \{\pi_1, \pi_2, \ldots, \pi_i\}$ for $i = 1, \ldots, |P|$ with $\bar{P}_0 = P$.
$R_{\boldsymbol{\pi}}(P) := P \setminus U_{\boldsymbol{\pi}}(P)$, where,

$$U_{\boldsymbol{\pi}}(P) := \left\{\pi_j \in P \mid \max_{i\in N\setminus P} \rho_i(\bar{P}_j) - \rho_{\pi_j}(\bar{P}_j) < \delta_{j-1}\right\}, \tag{14}$$

where $\delta_j = \delta_{j-1} - \left(\max_{i\in N\setminus P} \rho_i(\bar{P}_j) - \rho_{\pi_j}(\bar{P}_j)\right)^+$ and $\delta_0 = \delta > 0$



The extended pack inequality,

$$\mathbf{x}(N \setminus R_{\boldsymbol{\pi}}(P)) \ge |U_{\boldsymbol{\pi}}(P)| + 1 \tag{15}$$

is valid for X. Moreover (15) defines a facet of $conv(X^{R_{\boldsymbol{\pi}}(P)})$ if P is a maximal pack and $\forall\ i \in U_{\boldsymbol{\pi}}(P)\ \exists$ *distinct* $j_i, k_i \in N \setminus P$ such that,

$$f(P \cup \{j_i, k_i\} \setminus i) \ge d \tag{16}$$

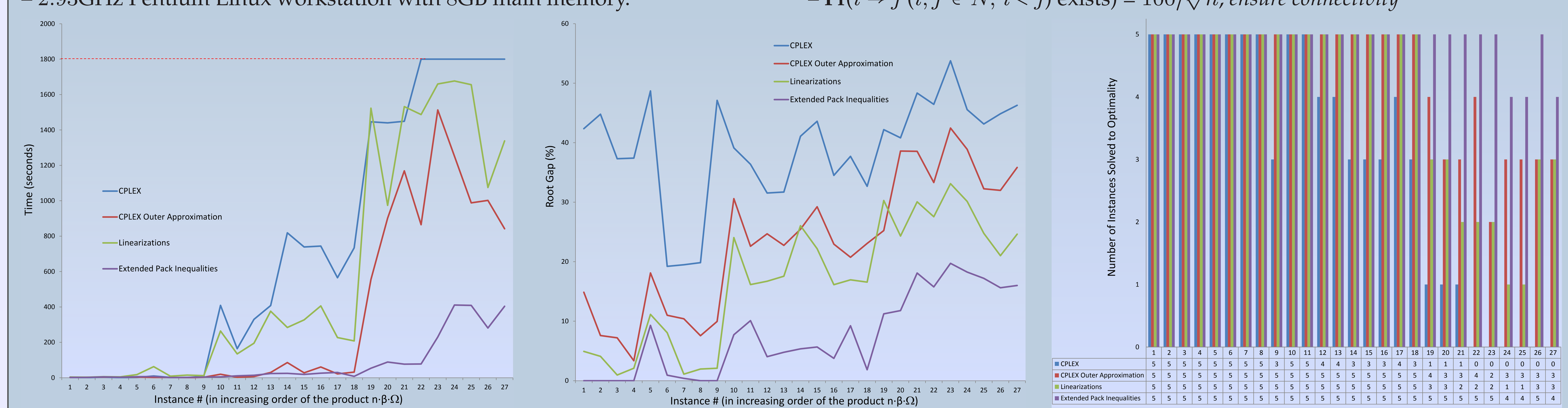
Computational Analysis

Setup

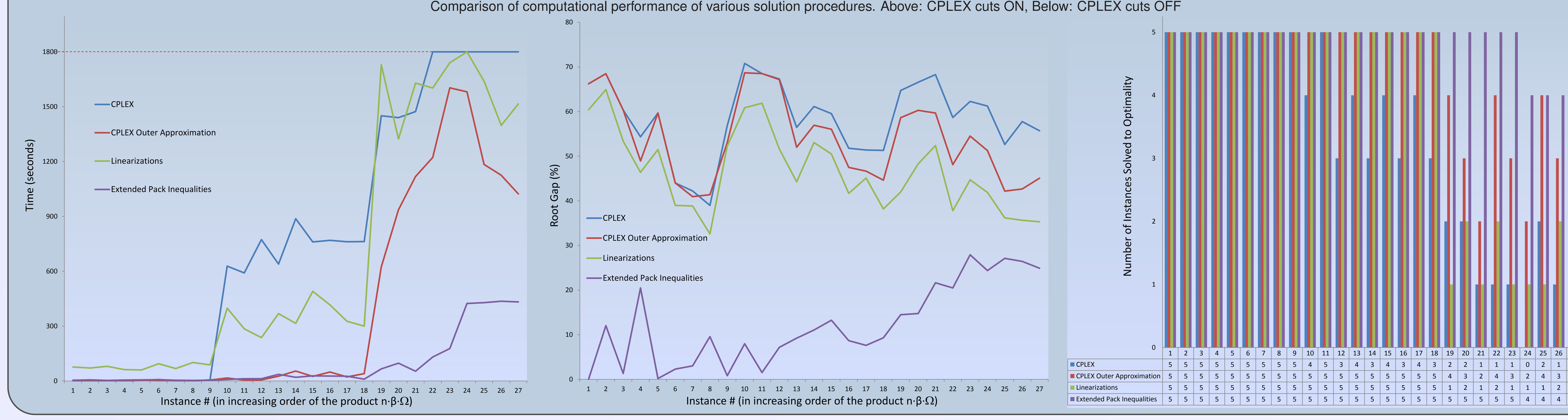
– MIP solver of CPLEX Version 12.3
– CPLEX heuristics are turned off, and a single thread is used.
– MIP search strategy is set to traditional *B & B*
– solver time limit 1800 secs. and memory limit 100MB.
– 2.93GHz Pentium Linux workstation with 8GB main memory.

Data generation

– $n \in \{10, 20, 40\}$, $\Omega \in \{1, 3, 5\}$ and $\beta \in \{0.3, 0.5, 0.7\}$
– observations averaged over 5 instances for each tuple (n, β, Ω).
– $d = \beta \cdot$ (maximum flow through N for given Ω),
– $u_i \sim U[100, 0]$ and $\sigma_i \sim U[0, u_i/\Omega]$
– $\mathbf{Pr}(i \to j\ (i, j \in N,\ i < j) \text{ exists}) = 100/\sqrt{n}$; *ensure connectivity*



Comparison of computational performance of various solution procedures. Above: CPLEX cuts ON, Below: CPLEX cuts OFF



Incorporating Correlations

• Issues

Separation of the conic inequalities, the objective of the separation problem no longer remains convex.

• Existing Literature

- Linearizations: Reformulation Linearization Technique [Adams, W.P. and Sherali, H.D., 1986], McCormick Inequalities [McCormick, G.P., 1976]
- Semidefinite/Second order cone relaxations: SDP and Cutting Planes [Helmberg. C, Rendl, F., 1998], Second order cone relaxations [Kim, S., Kojima, M.,2001]
- Convex Reformulations: [Hammer, P.L., Rubin, A.A., 1970], [Poljak, S., Rendl, F., Walkowicz, H., 1995], QCR method [Billionnet et. al, 2008]

• Sum of Squares reformulation

Let $\mathbf{Q} = [\,q_{ij}\,]_{1\le i,j\le n} \in \mathbb{R}^{n\times n}$ be any real $n \times n$ matrix map, $\mathbf{p} = [\,p_i\,]_{1\le i\le n} \in \mathbb{R}^n$ be any n dimensional real vector and $r \in \mathbb{R}$ be a constant. We define F and $\tilde{F}$ as,

$$F := \left\{\mathbf{x} \in [0,1]^n \mid \mathbf{x}'\,\mathbf{Q}\,\mathbf{x} + \mathbf{p}'\mathbf{x} + r \ge 0\right\} \tag{17}$$

$$\bar{F} := \left\{\mathbf{x} \in [0,1]^n : \sum_{i=1}^{n}\left[\sum_{\substack{j=1\\ j\ne i}}^{n}\left(q_{ij}^{+}\frac{(x_i - x_j)^2}{2} + q_{ij}^{-}\frac{(x_i + x_j)^2}{2}\right) + q_{ii}^{-}x_i^2\right] \le \sum_{i=1}^{n}\left[\sum_{\substack{j=1\\ j\ne i}}^{n}|q_{ij}|\left(\frac{x_i + x_j}{2}\right) + q_{ii}^{+}x_i\right] + \mathbf{p}'\mathbf{x} + r\right\} \tag{18}$$

where $q_{ij}^{+} = \max(0, q_{ij})$, $q_{ij}^{-} = -\min(0, q_{ij})$, $1 \le i, j \le n$ and $|\cdot| : \mathbb{R} \mapsto \mathbb{R}_+$ denotes the absolute value function.. Then,

(i) $F \subseteq \tilde{F}$
(ii) $F \cap \mathbb{Z}^n = \tilde{F} \cap \mathbb{Z}^n$

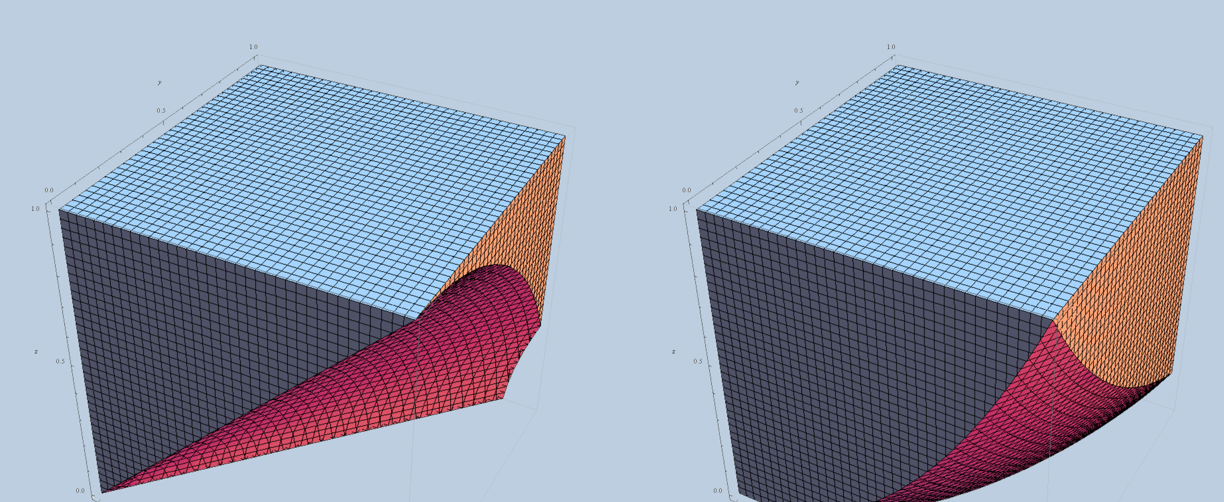


Figure: Non-convex set $x \le \sqrt{y^2 + z^2 - 0.2xy + 0.27yz}$ and the convex relaxation $0.135(y-z)^2 + 0.1(x+y)^2 + x^2 \le 0.1x + 1.235y + 1.135z$