Proposition 1. We show that there exists a polynomial time algorithm to check the closedness of the convex hull of a simple convex set.

Proposition 2. Second order mixed integer conic programming (SOCMIP) is said to be a lattice cone w.r.t to \( P \).

Definition 2. For a matrix \( (A, b, z) \in \mathbb{Z}^{m \times (n+m)} \), where for all \( i \), \( a_i \in \mathbb{Q}^{m \times n+k} \), \( b_i \in \mathbb{Q}^m \), and \( z_i \in \mathbb{R}^{m \times n+m} \), the Lorentz cone is the Lorentz cone in \( \mathbb{R}^m \).

Theorem 1. Let \( P = \{ (x,y) \in \mathbb{Z}^{m \times (n+m)} \mid Ax+By-b \in \mathbb{Z}^{m} \} \) and let \( V = \{ (Ax+By-b) \mid (x,y) \in \mathbb{R}^{m \times n+m} \} \). Then, \( \text{conv}(P \cap \mathbb{Z}^{m \times (n+m)}) \) is closed if and only if one of the following hold:

1. \( 0 \not\in \mathbb{L}^{(m \times n+m)} \).
2. \( 0 \in \mathbb{L}^{(m \times n+m)} \) and \( \dim(\mathbb{L}^{(m \times n+m)}) \leq 1 \).
3. \( 0 \in \mathbb{L}^{(m \times n+m)} \) and \( \dim(\mathbb{L}^{(m \times n+m)}) \geq 2 \).

2. Checking closedness in polynomial time

Theorem 2. There exists an algorithm that runs in polynomial time with respect to \( \mathbb{L}^{(m \times n+m)} \) to check whether \( \text{conv}(\mathbb{L}^{(m \times n+m)}) \) is closed.

3. Integer hulls of a more general class of (SOCMIP) problems. Consider the sets \( P = \{ (x,y) \in \mathbb{Z}^{m \times (n+m)} \mid Ax+By-b \in \mathbb{Z}^{m} \} \), where for all \( i \), \( a_i \in \mathbb{Q}^{m \times n+k} \), \( b_i \in \mathbb{Q}^m \), and \( z_i \in \mathbb{R}^{m \times n+m} \), the Lorentz cone is the Lorentz cone in \( \mathbb{R}^m \).

Theorem 3. There exists an algorithm that runs in polynomial time with respect to \( \text{max}(|P|) \) to check whether \( \text{conv}(\mathbb{L}^{(m \times n+m)}) \) is closed.

Proof Ideas

1. Characterization of closedness

Step 1: Simplifying the problem.

Using rationality of the data, we show that \( \text{conv}(\mathbb{L}^{(m \times n+m)} \cap V) \) is closed if and only if \( \text{dim}(\mathbb{L}^{(m \times n+m)} \cap V) \) is closed.

Step 2: Analyzing the set \( \mathbb{L}^{(m \times n+m)} \cap V \).

We have two cases:

- **Case 1:** If \( 0 \not\in \mathbb{L}^{(m \times n+m)} \cap V \), then \( \mathbb{L}^{(m \times n+m)} \cap V \) is a closed strictly convex set.

- **Case 2:** If \( 0 \in \mathbb{L}^{(m \times n+m)} \cap V \), then \( \mathbb{L}^{(m \times n+m)} \cap V \) is a mixed integer lattice.

- **Case 2a:** If \( 0 \in \mathbb{L}^{(m \times n+m)} \cap V \) and \( \mathbb{L}^{(m \times n+m)} \cap V \) is a closed convex cone.

- **Case 2b:** If \( 0 \in \mathbb{L}^{(m \times n+m)} \cap V \) and \( \mathbb{L}^{(m \times n+m)} \cap V \) is a two-dimensional cone.

By Proposition 2, we only need to check if the two extreme rays of \( \mathbb{L}^{(m \times n+m)} \cap V \) belong to the lattice \( \mathbb{L}^{(m \times n+m)} \).

Notation and Definitions.

Second order mixed integer conic programming (SOCMIP).

Let \( A \in \mathbb{Q}^{m \times n} \), \( B \in \mathbb{Q}^{m \times m} \) and \( b \in \mathbb{Q}^m \). The feasible region of a 'simple' second order mixed integer conic programming problem is given by the set

\[ P = \{ (x,y) \in \mathbb{Z}^{m \times (n+m)} \mid Ax+By-b \in \mathbb{L}^{m} \} \]

where \( \mathbb{L}^m = \{ v \in \mathbb{R}^m \mid \sum_{i=1}^m v_i^2 \leq u_0 \} \) is the Lorentz cone in \( \mathbb{R}^m \).

- **Definition 1:** Strictly convex sets. A set \( K \subseteq \mathbb{R}^d \) is called a strictly convex set, if for all \( K \) and for all \( x, y \in K, x \neq y \), \( \forall \alpha, \beta \in \mathbb{R} \), \( \alpha + \beta = 1 \), \( \alpha, \beta \geq 0 \), \( \alpha x + \beta y \not\in K \).

- **Definition 2:** Mixed integer lattice. Let \( A \in \mathbb{Q}^{m \times n} \) and \( B \in \mathbb{Q}^{m \times m} \). Then the set \( \{ x \in \mathbb{R}^m \mid Ax+By-b \in \mathbb{Z}^m \} \) is said to be the mixed integer lattice generated by \( P \).

- **Definition 3:** Lattice cone. \( \mathbb{L}^m \subset \mathbb{R}^m \) is a mixed integer lattice. A pointed cone \( K \subseteq \mathbb{R}^m \) is said to be a lattice cone if \( \mathbb{L}^m = \text{conv}(K) \).

- **Definition 4:** For a matrix \( M \), we denote \( M \) the linear subspace generated by its columns.

Preliminary Results

Mixed integer hulls of strictly convex sets

**Proposition 1.** Let \( K \subseteq \mathbb{R}^d \) be a closed strictly convex set and let \( b \in \mathbb{R}^n \). Then \( \text{conv}(K \cap \mathbb{Z}^d - b) \) is closed.

Mixed integer hulls of closed convex cones

**Proposition 2.** Let \( K \subseteq \mathbb{R}^d \) be a full-dimensional pointed closed convex cone. Then \( \text{conv}(K \cap \mathbb{Z}^d) - K \cap \mathbb{Z}^d \) is closed if and only if \( K \cap \mathbb{W}^d \) is a lattice cone with \( \mathbb{L}^d \).

Intersection of integer hulls

Assume that \( L \) does not have any continuous components, that is, \( \mathbb{L} = \mathbb{Z}^m \).

**Proposition 3.** Let \( K \subseteq \mathbb{R}^d \), \( i = 1, 2, \) be closed convex sets. Assume \( \text{conv}(K_i \cap \mathbb{L}^d) \) is closed for \( i = 1, 2 \). \( \text{dim}(K_1 \cap K_2) = r \) is a rational linear subspace, then \( \text{conv}(K_1 \cap K_2 \cap \mathbb{L}^d) \) is closed.

Remark: Unfortunately, Proposition 3 is not necessarily true for a general mixed integer lattice in the case \( n = 2 \).