Closedness of Mixed Integer Hulls of SOMICP

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Summary

We show that there exists a polynomial time algorithm to check the closedness of the convex hull of the feasible region of a simple class of second order conic mixed integer programming problems (SOCMIP). In the case of pure integer problems, we generalize this result for the intersection of simple SOCMIP's.

Notation and Definitions.

Second order mixed integer conic programming (SOCMIP)

Let $A \in \mathbb{Q}^{m\times n_1}$, $B \in \mathbb{Q}^{m\times n_2}$ and $b \in \mathbb{Q}^m$. The feasible region of a 'simple' second order conic mixed integer programming problem is given by the set

$$\mathcal{P} = \{(x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \mid Ax + By - b \in \mathbf{L}^m\},$$

where $\mathbf{L}^m = \{u \in \mathbb{R}^m \mid \sqrt{\sum_{i=1}^{m-1} u_i^2} \leq u_m\}$ is the Lorentz cone in $\mathbb{R}^m$.

- Let us define the size of the problem $\mathcal{P}$ as $\text{size}(\mathcal{P}) = \text{size}(A) + \text{size}(B) + \text{size}(b)$.
- We denote the translated mixed integer lattice defined by $\mathcal{P}$ as

$$\mathcal{L}^{\mathcal{P}} = \{x \in \mathbb{R}^m \mid x = Az + By - b,\ z \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}\}.$$

Others

Definition 1 (Strictly convex set)**.** *A set $K \subseteq \mathbb{R}^n$ is called a* strictly convex set, *if K is a convex set and for all $x, y \in K$, $\lambda x + (1-\lambda)y \in \text{rel.int}(K)$ for $\lambda \in (0, 1)$.*

Definition 2 (Mixed integer lattice)**.** *Let $A \in \mathbb{Q}^{m\times n_1}$ and $B \in \mathbb{Q}^{m\times n_2}$. Then the set*

$$\{x \in \mathbb{R}^m \mid x = Az + By,\ z \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}\}$$

is said to be the mixed integer lattice generated by A and B.

Definition 3 (Lattice cone)**.** *Let $\mathcal{L}' \subseteq \mathbb{R}^n$ be a mixed integer lattice. A pointed cone K is said to be a lattice cone w.r.t to $\mathcal{L}'$ if all the extreme rays of K can be scaled to belong to $\mathcal{L}'$.*

- Let $\text{conv}(K)$ denotes the convex hull of a set $K \subseteq \mathbb{R}^n$.
- Let $\mathcal{L}' \subseteq \mathbb{R}^m$ denote an arbitrary mixed integer lattice.
- For a matrix M, we denote $\langle M\rangle$ the linear subspace generated by its columns.

Preliminary Results

Mixed integer hulls of strictly convex sets

Proposition 1. *Let $K \subseteq \mathbb{R}^m$ be a closed strictly convex set and let $b \in \mathbb{R}^n$. Then $\text{conv}(K \cap [\mathcal{L}' + b])$ is closed.*

Mixed integer hulls of closed convex cones

Proposition 2. *Let $K \subset \mathbb{R}^m$ be a full-dimensional pointed closed convex cone. Then $\overline{\text{conv}}(K \cap \mathcal{L}') = K \cap W$, where $W = \text{aff}(K \cap \mathcal{L}')$. Moreover, $\text{conv}(K \cap \mathcal{L}')$ is closed if and only if $K \cap W$ is a lattice cone w.r.t $\mathcal{L}'$.*

Intersection of integer hulls

Assume that $\mathcal{L}'$ does not have any continuous components, that is, $n_2 = 0$.

Proposition 3. *Let $K_i \subseteq \mathbb{R}^m$, $i = 1, 2$, be closed convex sets. Assume $\text{conv}(K_i \cap \mathcal{L}')$ is closed for $i = 1, 2$. If $\text{lin.space}(K_1 \cap K_2)$ is a rational linear subspace, then $\text{conv}\left[(K_1 \cap K_2) \cap \mathcal{L}'\right]$ is closed.*

Remark: Unfortunately, Proposition 3 is not necessarily true for a general mixed integer lattice in the case $n_2 > 0$.

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Main Results

1. Characterization of closedness

Theorem 1. *Let $\mathcal{P} = \{(x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \mid Ax + By - b \in \mathbf{L}^m\}$ and let $V = \{Ax + By - b \mid (x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\}$. Then $\text{conv}\,[\mathcal{P} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})]$ is closed if and only if one of the following hold*

1. $0 \notin \mathbf{L}^m \cap V$.
2. $0 \in \mathbf{L}^m \cap V$ *and* $\dim(\mathbf{L}^m \cap V) \leq 1$.
3. $0 \in \mathbf{L}^m \cap V$, $\dim(\mathbf{L}^m \cap V) = 2, n_2 = 0$ *and* $\mathbf{L}^m \cap V$ *is a lattice cone w.r.t.* $\mathcal{L}^{\mathcal{P}}$.
4. $0 \in \mathbf{L}^m \cap V$, $\dim(\mathbf{L}^m \cap V) \geq 2$ *and* $\dim(\langle B\rangle) \geq \dim(V) - 1$.

2. Checking closedness in polynomial time

Theorem 2. *There exists an algorithm that runs in polynomial time with respect to $\text{size}(\mathcal{P})$ to check whether $\text{conv}(\mathcal{P} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}))$ is closed.*

3. Integer hulls of a more general class of (SOCMIP) problems.

Consider the sets $\mathcal{P}_i = \{x \in \mathbb{Z}^n \mid A_i x - b_i \in \mathbf{L}^{m_i}\}$, where for all $i = 1, \ldots, q$, we have $A_i \in \mathbb{Q}^{m_i\times n}$, $b_i \in \mathbb{Q}^{m_i}$, and $\mathbf{L}^{m_i} \subseteq \mathbb{R}^{m_i}$ is the Lorentz cone in $\mathbb{R}^{m_i}$.

Theorem 3. *There exists an algorithm that runs in polynomial time with respect to $\max\{\text{size}(\mathcal{P}_i) \mid i = 1, \ldots, q\}$ to check whether $\text{conv}(\bigcap_{i=1}^{q} \mathcal{P}_i)$ is closed.*

Proof Ideas

1. Characterization of closedness

Step 1: Simplifying the problem.

Using rationality of the data, we show that

$$\text{conv}(\mathcal{P} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})) \text{ is closed} \Leftrightarrow \text{conv}((\mathbf{L}^m \cap V) \cap \mathcal{L}^{\mathcal{P}}) \text{ is closed}$$

Step 2: Analyzing the set $(\mathbf{L}^m \cap V)$.

We have two cases:

- **Case 1:** If $0 \notin (\mathbf{L}^m \cap V)$, then
  - $(\mathbf{L}^m \cap V)$ is a closed strictly convex set.

$\Rightarrow$ By Proposition 2, we obtain that $\text{conv}((\mathbf{L}^m \cap V) \cap \mathcal{L}^{\mathcal{P}})$ is closed.

Example 1. *Illustration of Case 1*

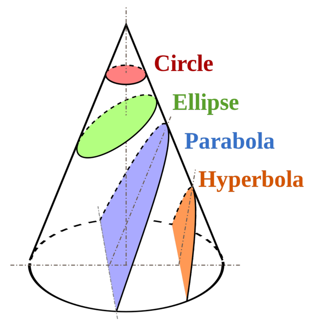

Fig. 1

- **Case 2:** If $0 \in (\mathbf{L}^m \cap V)$, then
  - $\mathcal{L}^{\mathcal{P}}$ is a mixed integer lattice.
  - $(\mathbf{L}^m \cap V)$ is a pointed closed convex cone.
  - Based on $\dim((\mathbf{L}^m \cap V))$, we have 3 subcases.
- **Case 2a:** If we have $\dim(\mathbf{L}^m \cap V) \leq 1$, then
  - The cone $\mathbf{L}^m \cap V$ is just the zero vector or a ray.

$\Rightarrow$ Very easy case: it is always closed!

- **Case 2b:** If we have $\dim(\mathbf{L}^m \cap V) = 2$, $\text{rank}(B) = 0$, then
  - $\mathcal{L}^{\mathcal{P}} = \{Ax + By \mid x \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}\} = \{Ax \mid x \in \mathbb{Z}^{n_1}\}$.
  - $(\mathbf{L}^m \cap V)$ is a two dimensional cone.

$\Rightarrow$ By Proposition 2, we only need to check if the two extreme rays of $\mathbf{L}^m \cap V$ belongs to the lattice $\mathcal{L}^{\mathcal{P}}$.

Source Fig. 1 http://commons.wikimedia.org/wiki/File:Secciones_c\%C3\%B3nicas.svg
Source Fig. 2,3 DAMR, SSD.

- **Case 2c:** If we have $\dim(\mathbf{L}^m \cap V) \geq 3$, then
  - In order to use Proposition 2 we need the following lemma.

    Lemma 4. *Assume that $0 \in \mathbf{L}^m \cap V$ and that $[A\ B] \in \mathbb{Q}^{m\times n}$. Then*
    1. *Let $\dim(\mathbf{L}^m \cap V) = 2$. If $\text{rank}(B) \geq \dim(V) - 1$, then $\mathbf{L}^m \cap V$ is a lattice cone w.r.t. $\mathcal{L}^{\mathcal{P}}$.*
    2. *Let $\dim(\mathbf{L}^m \cap V) \geq 3$. Then $\text{rank}(B) \geq \dim(V) - 1$ if and only if $\mathbf{L}^m \cap V$ is a lattice cone w.r.t. $\mathcal{L}^{\mathcal{P}}$.*
  - Lemma 4 says that $\text{conv}((\mathbf{L}^m \cap V) \cap \mathcal{L}^{\mathcal{P}})$ is closed when there are 'sufficiently many continuous variables'.

Example 2. $\mathcal{L}^{\mathcal{P}} = \mathbb{Z}^2 \times \mathbb{R}^1$ *(not closed)*

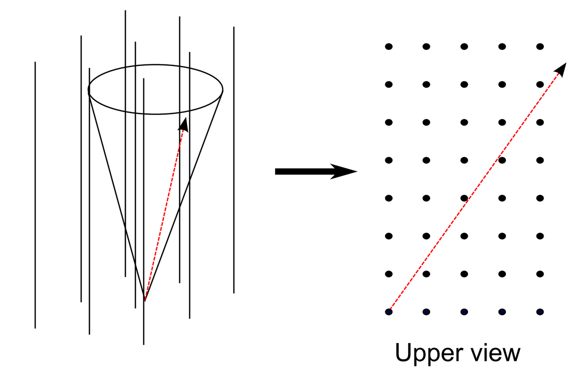

Fig. 2

Example 3. $\mathcal{L}^{\mathcal{P}} = \mathbb{Z}^1 \times \mathbb{R}^2$ *(closed)*

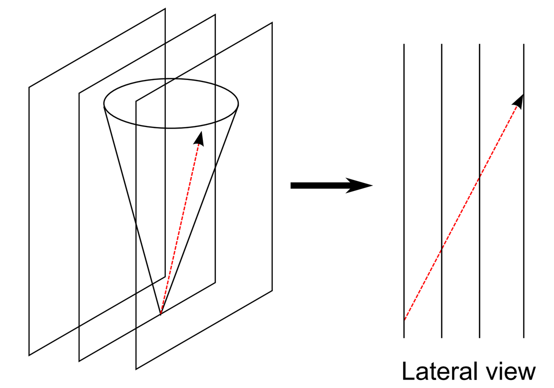

Fig. 3

$\Rightarrow$ By Lemma 4 and Proposition 2 we only need to check if $\text{rank}(B) \geq \dim(V) - 1$.

2. Checking closedness in polynomial time

We only need to verify that the conditions given by Theorem 1 can be checked in polynomial time.

Step 1 Check if $0 \in \mathbf{L}^m \cap V$.

- $0 \in \mathbf{L}^m \cap V$ if and only if $b \in \text{span}([A\ B])$.
- We only need to solve a linear system with rational data.

Step 2 Compute $\dim(\mathbf{L}^m \cap V)$, $\dim(V)$ and $\text{rank}(B)$, when $0 \in \mathbf{L}^m \cap V$.

- Let Proj_V denote the orthogonal projection over the linear subspace V. We need the following lemma.

  Lemma 5.
  1. $\dim(\text{int}(\mathbf{L}^m) \cap V) \leq 1$ *if and only if* $\text{int}(\mathbf{L}^m) \cap V = \emptyset$ *or* $\dim(V) \leq 1$.
  2. *Lets denote* $a := (0, 1) \in \mathbb{R}^{m-1} \times \mathbb{R}$. *Then*

  $$\text{int}(\mathbf{L}^m) \cap V \neq \emptyset \text{ if and only if } \text{Proj}_V(a) \in \text{int}(\mathbf{L}^m).$$

  - We can check whether $\text{int}(\mathbf{L}^m) \cap V = \emptyset$ by computing $\text{Proj}_V(a)$.
  - If $\dim(\mathbf{L}^m \cap V) \geq 2$, then $\dim(V) = \dim(\mathbf{L}^m \cap V)$.
  - Since B is rational, $\text{rank}(B)$ can be computed in polynomial time.

Step 3 Verifying if $\mathbf{L}^m \cap V$ is a lattice cone w.r.t. $\mathcal{L}^{\mathcal{P}}$.

- We want to check if the two extreme rays of $\mathbf{L}^m \cap V$ belongs to the lattice $\mathcal{L}^{\mathcal{P}} = \{Ax \mid x \in \mathbb{Z}^{n_1}\}$ $(*)$
  - Find a basis $\{p, q\} \subseteq \mathbb{R}^m$ of the lattice $\mathcal{L}^{\mathcal{P}}$, by using the Hermite Normal form algorithm.
  - $(*)$ is satisfied if and only if the following system has rational solutions in (α, β):

$$\sum_{i=1}^{m-1} (\alpha p_i + \beta q_i)^2 = 1, \quad \alpha p_m + \beta q_m = 1. \tag{1}$$

  - (1) reduces to determining if a quadratic equation with integer data has rational roots.
  - We need to verify if the discriminant of the quadratic equation is a perfect square.
  - Checking whether an integer is a perfect square can be done in polynomial time using well-know algorithms.

3. Integer hulls of a more general class of SOCP problems.

This is a direct consequence of Theorem 1 and Proposition 3.