Improved optimality cuts for the integer L-shaped method

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Summary

The integer L-shaped method is a well-known approach to two-stage stochastic programs having binary first-stage decision variables and mixed-integer recourse. It basically relies on the generation of optimality cuts to approximate the recourse function. However, these cuts may be weak as they only use local information of the solution to be cut off, and therefore many of them may be needed to solve a given problem. In this work, we attempt to generate optimality cuts based on the information provided by all feasible solutions visited so far at a given step of the method. Our hope is that this modification yields a reduction in the number of cuts needed and in the overall computation time.

Introduction

Consider a two-stage stochastic integer programming problem of the form

$$\begin{aligned} \min \quad & c^\top x + Q(x) \\ \text{s.t.} \quad & Ax = b \\ & x \in X \subseteq B := \{0,1\}^n, \end{aligned}$$

where $Q(x) = \mathbb{E}_\xi[\min_y\{q^\top y :\ Wy = h - Tx,\ y \in Y \subseteq \mathbb{R}^m_+\}]$ and $\xi = (q, T, h)$.

Assumptions:
- Y enforces some integrality requirements on y.
- Given a binary first stage decision vector x, the function $Q(x)$ is computable from x.
- There exists a finite value L satisfying $L \leqslant \min_x\{Q(x) :\ Ax = b,\ x \in B\}$.

To solve this class of problems, the integer L-shaped method [1] solves a master problem of the form

$$\begin{aligned} \min \quad & c^\top x + \theta \\ \text{s.t.} \quad & Ax = b \\ & D_k x \geqslant d_k \quad k = 1, \dots, s && (1) \\ & E_k x + \theta \geqslant e_k \quad k = 1, \dots, t && (2) \\ & x \geqslant 0,\ \theta \in \mathbb{R}. \end{aligned}$$

- Feasibility cuts (1) are used to ensure $Q(x) \in \mathbb{R}$.
- Optimality cuts (2) are used to express θ as an appropriate approximation of $Q(x)$.
- These cuts are generated dynamically through the procedure.

Let $r = 1, \dots, R$ index the feasible solutions. Let $x_i = 1$, $i \in S_r$, and $x_i = 0$, $i \notin S_r$, be the r-th feasible solution, and θ_r the corresponding expected second-stage value. In [1], the optimality cut is defined as

$$\theta \geqslant (\theta_r - L)\left(\sum_{i \in S_r} x_i - \sum_{i \notin S_r} x_i\right) - (\theta_r - L)(|S_r| - 1) + L. \quad (3)$$

Note that (3) is defined in terms of the r-th feasible solution x^r **only** and its expected second-stage value θ_r. Moreover, the right-hand-side takes the value θ_r at this particular point, and a value less or equal than L otherwise.

New optimality cuts

At a given stage of the integer L-shaped method, let $V \subseteq \{1, \dots, R\}$ be the set of feasible solutions visited so far. We would like to design an optimality cut of the form

$$\theta \geqslant \alpha^\top x + \beta \quad (4)$$

that takes into account the available information provided by **all** the solutions in V. To that end, consider the cut generating problem

$$\begin{aligned} (CGP) \quad \min_{\alpha,\beta,\eta} \quad & \eta \\ \text{s.t.} \quad & \theta_r \geqslant \alpha^\top x^r + \beta \quad \forall r \in V && (5) \\ & L \geqslant \alpha^\top x^r + \beta \quad \forall r \notin V && (6) \\ & \eta \geqslant \theta_r - (\alpha^\top x^r + \beta) \quad \forall r \in V && (7) \\ & \eta \geqslant L - (\alpha^\top x^r + \beta) \quad \forall r \notin V. && (8) \end{aligned}$$

- Constraints (5) and (6) ensure that $\alpha^\top x + \beta$ is a lower bound on $Q(x)$.
- The objective is to minimize the maximum difference between the best current lower bound and the lower bound provided by (4) among all solutions.
- Given that V is explicitly known, (5) and (7) pose no difficulty.
- However, (6) and (8) may involve exponentially many constraints which can be equivalently written as

$$L \geqslant \max\{\alpha^\top x^r + \beta :\ r \notin V\} = \max\{\alpha^\top x + \beta :\ x \in conv(B \backslash X_V)\}$$

$$L - \eta \leqslant \min\{\alpha^\top x^r + \beta :\ r \notin V\} = \min\{\alpha^\top x + \beta :\ x \in conv(B \backslash X_V)\},$$

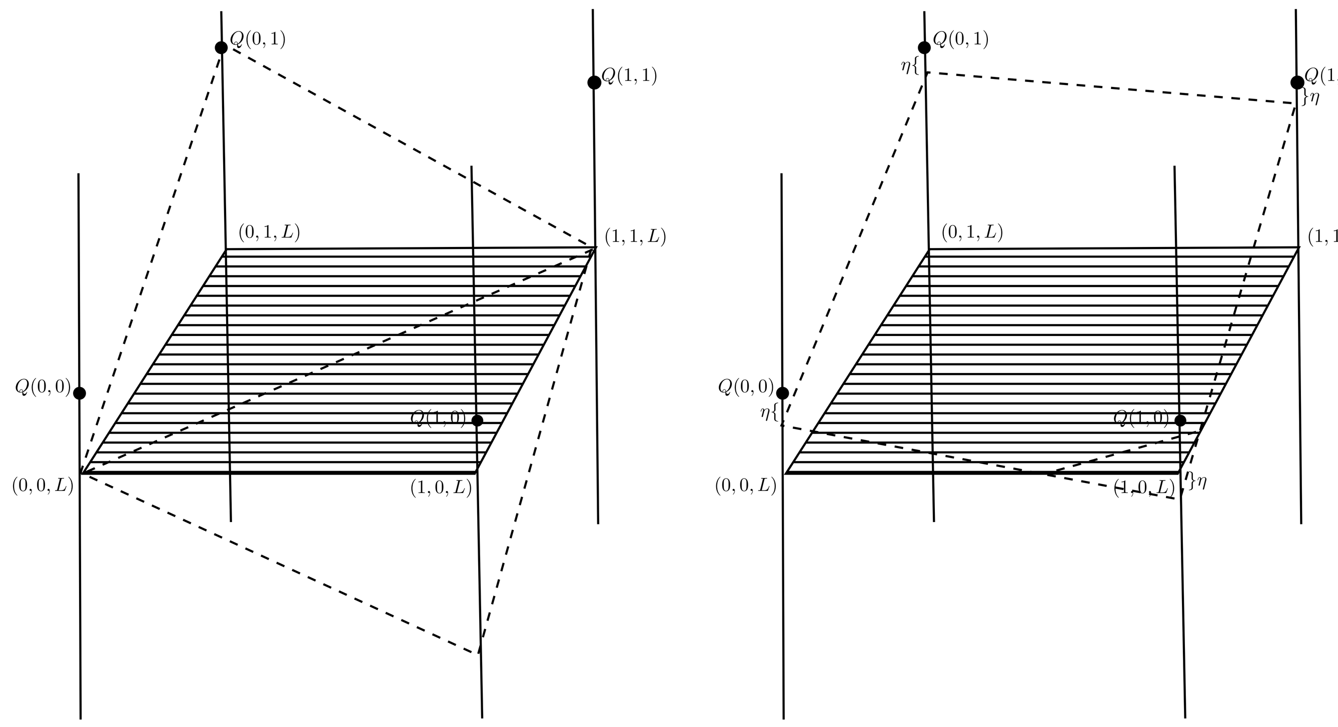
where $X_V := \{x^r \in B :\ r \in V\}$. Therefore, the tractability of (CGP) depends upon the tractability of $conv(B \backslash X_V)$. Fortunately, we prove the following result which allows to recast (CGP) as an LP.

Theorem 1 *Given explicit V, $conv(B \backslash X_V)$ admits an extended formulation having $\mathcal{O}(n|V|)$ variables and constraints.*

A key step in proving Theorem 1 is the following proposition which is a two-sided extension of the superincreasing coefficient knapsack studied in [2].

Proposition 1 *Given integers $a \leqslant b$, $conv(\{x \in B :\ a \leqslant \sum_{i=1}^n 2^{i-1} x_i \leqslant b\})$ has a complete linear description having $\mathcal{O}(n)$ constraints.*

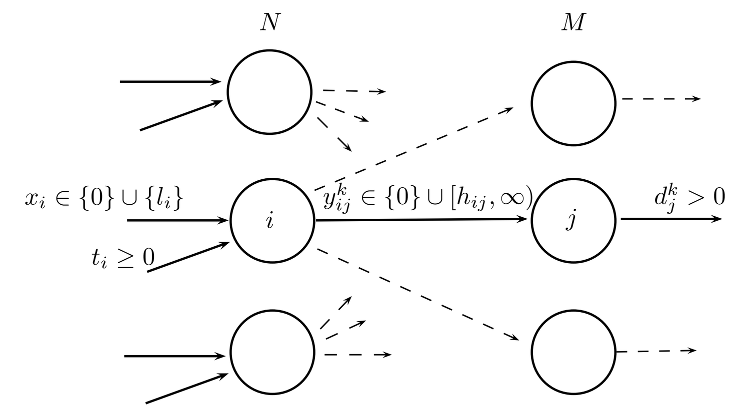
Notice that conceptually, L can be replaced in (CGP) by any lower bounding function $L(x)$ such that $L(x) \leqslant Q(x)$, and in such case the cuts generated are expected to be stronger than when L is constant. Unfortunately, in this case (CGP) becomes intractable in general, but it can be casted as an IP problem when L is convex and piecewise affine. Also, feasibility cuts can be derived from separation over $conv(B \backslash X_V)$.



(a) Standard optimality cut (b) New optimality cut

Experiments

Consider the stochastic semi-continuous transportation problem depicted below.



The above problem can be formulated with the aid of binary variables.

$$\min \left\{ f^\top x + \sum_k \pi_k q^{k\top} y^k :\ \begin{array}{ll} \sum_i y^k_{ij} \geqslant d^k_j & \forall j, k \\ \sum_j y^k_{ij} \leqslant t_i + l_i x_i & \forall i, k \\ h_{ij} z^k_{ij} \leqslant y^k_{ij} \leqslant (t_i + l_i) z^k_{ij} & \forall i, j, k \\ x_i \in \{0, 1\} & \forall i \\ z^k_{ij} \in \{0, 1\} & \forall i, j, k. \end{array} \right\}$$

We compare the standard integer L-shaped method with an implementation which also includes the new optimality cuts.

	L-shaped		Combined L-shaped		
	Total Time	Nodes	Total Time	Partial Time	Nodes
1	173.77	1911	31.78	23	474
2	106.3	1344	195.93	64.29	942
3	220.49	2024	909.16	173.17	2099
4	189.91	1822	95.7	39.16	774
5	166.27	1560	48.75	41.95	898
6	344.03	3143	197.61	117.37	1936
7	612.09	5811	482.33	128.54	2666
8	350.33	3465	257.99	190.66	3016
9	340.67	2996	107.87	84.96	1588
10	286.37	2680	1806.09	124.68	2173
Avg.	279.02	2675.60	413.32	98.78	1656.60
Avg*	278.21	2675.11	258.57	95.90	1599.22

Table 1: $|N| = 20$, $|M| = 20$, $|K| = 30$. Avg* excludes instance 10.

	L-shaped		Combined L-shaped		
	Total Time	Nodes	Total Time	Partial Time	Nodes
1	511.62	1800	25.83	24.51	350
2	454.44	1688	340.6	193.4	853
3	1091.75	2717	668.05	390.06	2141
4	717.47	2117	288.06	217.06	1224
5	556.41	1666	117.94	112.14	668
6	1536.21	3882	717.2	397.4	2386
7	1800.56	5224	1802.55	575.31	3462
8	1332.09	3639	480.77	404.58	2398
9	1638.35	4360	1022.47	633.31	3228
10	1434.48	3590	1801.36	499.58	2965
Avg.	1107.34	3068.30	726.48	344.74	1967.50
Avg*	870.93	2429.89	406.77	263.61	1472.00

Table 2: $|N| = 20$, $|M| = 20$, $|K| = 100$. Avg* excludes instances 7 and 10.

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References

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[2] M. Laurent and A. Sassano. A characterization of knapsacks with the max-flow-min-cut property. Operations Research Letters 11, 2:105–110, 1992.