Cardinality-Constrained Continuous Mixing Set Arising in Joint Chance-Constrained Programming

Minjiao Zhang, Simge Küçükyavuz

Integrated Systems Engineering, The Ohio State University

Background

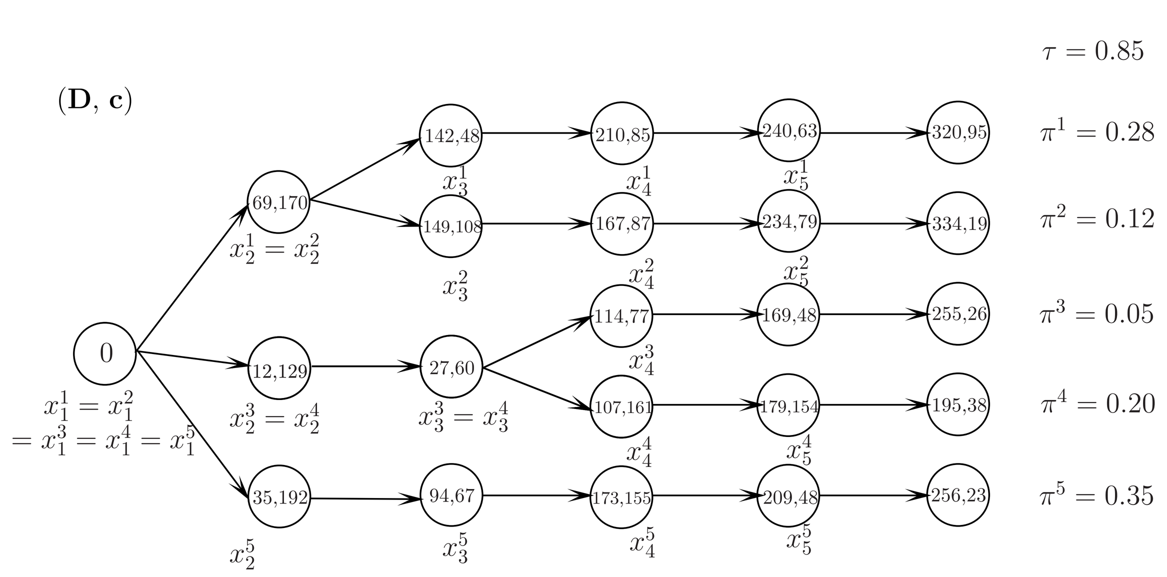
- **Joint chance-constrained problem**:
  - Finite-horizon dynamic (multi-stage) decision-making problem.
  - Uncertain data.
  - Risk / reliability / service level constraints.
- **Parameters**:
  - n: number of stages,
  - $[i,j]$: denotes the set $\{t \in \mathbb{Z} : i \le t \le j\}$,
  - π^i: probability of scenario i, $0 \le \pi^i \le 1$, $i \in [1,m]$,
  - τ: threshold reliability level, $0 \le \tau \le 1$,
  - $\Gamma = (\xi, \mu)$: random vector with finitely many realizations (scenarios),
  - $\Gamma_{t-1} := (\xi_1, \dots, \xi_{t-1}, \mu_1, \dots, \mu_{t-1})$: uncertainty revealed up to stage t, $t \in [2,n]$,
  - $A = \begin{pmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ A_{21} & A_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & A_{n3} & \cdots & A_{nn} \end{pmatrix}$: technology matrix.
- **Variables**:
  - x_1: decision at the first stage,
  - $x_t(\Gamma_{t-1})$: decision vector at stage $t \in [2,t]$ dependent of Γ_{t-1} observed.
- **Order of decisions**:

  $x_1 \to \Gamma_1 \to x_2(\Gamma_1) \to \Gamma_2 \to x_3(\Gamma_2) \to \cdots \to \Gamma_{n-1} \to x_n(\Gamma_{n-1})$
- **Dynamic joint chance-constrained program (DJCCP)**:

$$\min \mathbb{E}_{(\xi,\mu)}\{\mu^T x : \mathbb{P}\left[A \begin{pmatrix} x_1 \\ x_2(\Gamma_1) \\ \vdots \\ x_n(\Gamma_{n-1}) \end{pmatrix} \ge \xi \right] \ge \tau, x \in X\}.$$

Scenario-Based Model

- **Additional parameters**:
  - m: number of scenarios,
  - $D^i = (D^i_1, \dots, D^i_n)$: realization of ξ in scenario i, $i \in [1,m]$,
  - $c^i = (c^i_1, \dots, c^i_n)$: realization of μ in scenario i, $i \in [1,m]$.
- **Scenario tree**:



- **Variables**:
  - $x^i = (x^i_1, \dots, x^i_n)$: decisions made over the entire horizon under scenario i, $i \in [1,m]$.
- **Non-anticipativity**:

  $S^\ell_t = \{k \in [1,m] : D^\ell_j = D^k_j, c^\ell_j = c^k_j, j \in [1, t-1]\}$.
- **Deterministic equivalent of DJCCP**:

$$\begin{aligned} \min \quad & \textstyle\sum_{i=1}^m \sum_{t=1}^n \pi^i c^i_t x^i_t \\ \text{s.t} \quad & \textstyle\sum_{i=1}^t A_{ti} x^\ell_i \ge D^\ell_t (1 - z^\ell) \quad t \in [1,n], \ell \in [1,m], && (1) \\ & \textstyle\sum_{i=1}^m \pi^i z^i \le 1 - \tau, && (2) \\ & x^\ell_t = x^k_t \quad t \in [1,n], \ell \in [1,m], k \in S^\ell_t \setminus \{\ell\}, \\ & x^i \in X, z^i \in \{0,1\} \quad i \in [1,m]. \end{aligned}$$

Substructures of DJCCP

- **Mixing set**

  $K = \{(s,z) \in \mathbb{R}_+ \times \mathbb{Z}^n : s + h_i z_i \ge h_i, i = 1, \dots, n\}$
  - Inequalities (1) define a mixing set at each non-anticipative node.
  - Atamtürk et al. (2000), Günlük and Pochet (2001), Luedtke et al. (2010), Küçükyavuz (2012) propose valid inequalities for mixing set and its extensions.
- **Continuous mixing set (CMIX)**

  $U_n = \{(s,r,z) \in \mathbb{R} \times \mathbb{R}^n_+ \times \mathbb{Z}^n : s + r_i + z_i \ge f_i, i = 1, \dots, n\}$, where $1 > f_1 \ge f_2 \ge \cdots \ge f_n \ge 0$.

  Proposition 1 (Zhang et al. (2012)) For stages $1 \le t < T \le n$, scenario $\ell \in [1,m]$, a known lower bound $\bar{D}^\ell_t$ of $\bar{s} := \sum_{j=1}^t A_{Tj} x^\ell_j$, $R^\ell_t \subset \{k \in S^\ell_t \neq \emptyset : D^k_T \ge \bar{D}^\ell_t\}$, let $y^i_T := \sum_{j=1}^T A_{Tj} x^\ell_j$, the set

$$\begin{aligned} \bar{Q}^\ell_{tT} &= \{(\bar{s}, \{y^i_T\}_{i \in R^\ell_t}, \{z^i\}_{i \in R^\ell_t}) \in \mathbb{R} \times \mathbb{R}^{|R^\ell_t|}_+ \times \{0,1\}^{|R^\ell_t|} : \\ &\qquad y^i_T + D^i_T z^i \ge D^i_T, y^i_T \ge \bar{s} \ge \bar{D}^\ell_t, i \in R^\ell_t\} \\ &\equiv \{(\bar{s}, \{y^i_T\}_{i \in R^\ell_t}, \{z^i\}_{i \in R^\ell_t}) \in \mathbb{R} \times \mathbb{R}^{|R^\ell_t|}_+ \times \{0,1\}^{|R^\ell_t|} : \\ &\qquad \frac{(\bar{s} - \bar{D}^\ell_t)}{\bar{D}^\ell_{tT}} + \frac{y^i_T - \bar{s}}{\bar{D}^\ell_{tT}} + z^i \ge \frac{(D^i_T - \bar{D}^\ell_t)}{\bar{D}^\ell_{tT}}, y^i_T \ge \bar{s} \ge \bar{D}^\ell_t, i \in R^\ell_t\}. \end{aligned}$$

  is a substructure of the deterministic equivalent of DJCCP, where $\bar{D}^\ell_{tT} := \max\{D^i_T : i \in R^\ell_t\} - \bar{D}^\ell_t$; and set $\bar{Q}^\ell_{tT}$ is a continuous mixing set for which cycle inequalities of van Vyve (2005) are applicable.
- **Cardinality-constrained continuous mixing set (CCMIX)**

  $Q^k_n = \{(s,r,z) \in \mathbb{R} \times \mathbb{R}^n_+ \times \{0,1\}^n : s + r_i + z_i \ge f_i, i = 1, \dots, n, \sum_{j=1}^n z_j \le k\}$, where $0 \le k \le n$, $1 > f_1 \ge \cdots \ge f_n \ge 0$.
  - Knapsack inequality (2) can be relaxed to a cardinality constraint (extended cover inequality, cf. Wolsey (1998)).
  - Cardinality-constrained continuous mixing set is a relaxation of the deterministic equivalent of DJCCP.

Valid Inequalities of CCMIX

- **Extreme rays of CCMIX** $\{(0, e_j, \underline{0})\}^n_{j=1}$, $(1, \underline{0}, \underline{0})$, $(-1, \underline{1}, \underline{0})$
- **Extreme points of CCMIX**

  $T := \{t_1, \dots, t_l\} \subseteq \{1, \dots, n\}$, where $1 \le l \le k$, $f_{t_i} \ge f_{t_{i+1}}$ for $i = 1, \dots, l-1$. For a given set T, define point $P^T_i = (\bar{s}, \bar{r}, \bar{z})$ as:
  - if $i \in T$, $\bar{s} = f_i - 1$, and for $j \in \{1, \dots, n\}$,

$$(\bar{r}_j, \bar{z}_j) = \begin{cases} ((f_j - f_i)^+, 1) & \text{if } j \in T, \\ ((f_j - f_i + 1), 0) & \text{if } j \notin T; \end{cases}$$

  - if $i \notin T$,, $\bar{s} = f_i$, and for $j \in \{1, \dots, n\}$,

$$(\bar{r}_j, \bar{z}_j) = \begin{cases} (0,1) & \text{if } j \in T, \\ ((f_j - f_i)^+, 0) & \text{if } j \notin T. \end{cases}$$

  Extreme points of $\mathrm{conv}(Q^k_n)$ are $\cup_{T:0 \le |T| \le k} \{P^T_i\}^n_{i=1}$.
- **A class of valid inequalities**

  Let $p \in \{k, \dots, n\}$, $T = \{t_1, \dots, t_k\} \subseteq \{1, \dots, p\}$ with $f_{t_1} \ge \cdots \ge f_{t_k}$, $H = \{h_1, \dots, h_a\} \subseteq \{p+1, \dots, n\}$ with $f_{h_1} \ge \cdots \ge f_{h_a}$, where $a = 0$ if $H = \emptyset$, $a \in [1, n-p]$ otherwise. Inequality

$$(k + a - 1)s + \sum_{i=1}^k (r_{t_i} + \beta_{t_i} z_{t_i}) + \sum_{i=1}^a (r_{h_i} + (k - 1 + f_{h_i} - f_{h_a} - \lambda_{h_i}) z_{h_i}) \ge \sum_{i=1}^k f_{t_i} + \sum_{i=1}^{a-1} f_{h_i}$$

  is valid for Q^k_n, where

$$\beta_{t_i} = \begin{cases} f_{t_1} - f_{h_a} & \text{if } i = 1, \\ f_{t_i} - f_{t_{i-1}} + 1 & \text{if } i = 2, \dots, k, \end{cases}$$

  $\lambda_{h_i} = 0$ if $k = 1$, otherwise, λ_{h_i} is the sum of the smallest $k-1$ numbers in $\{\beta_{t_j} : j = 1, \dots, k\} \cup \{k - 1 + f_{h_j} - f_{h_a} - \lambda_{h_j} : j = 1, \dots, i-1\}$.
- We give necessary and sufficient conditions for inequalities to be valid for $\mathrm{conv}(Q^k_n)$.

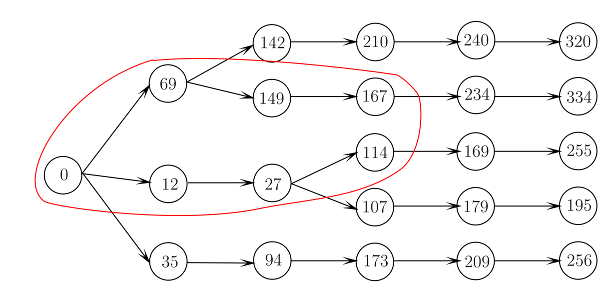
Examples

Let $t = 1$, $\ell = 1$, $T = 3$, consider the constraints associated with $R^\ell_t = \{2,3\} \subseteq S^1_1 = \{1,2,3,4,5\}$:

$x^2_1 + x^2_2 + x^2_3 \ge 167(1 - z^2)$,

$x^3_1 + x^3_2 + x^3_3 \ge 114(1 - z^3)$.

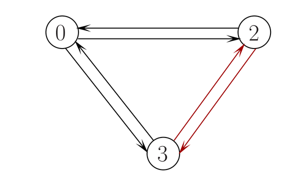
Note $x^2_1 = x^3_1$ and $\bar{D}^1_1 = 69$. $\bar{D}^1_{1,2} = \max\{167, 114\} - \bar{D}^1_1 = 98$.



Then

$$\frac{x^2_1 - 69}{98} + \frac{(x^2_2 + x^2_3)}{98} + z^2 - 1 \ge 0,$$

$$\frac{x^3_1 - 69}{98} + \frac{(x^3_2 + x^3_3)}{98} + z^3 \ge \frac{45}{98}.$$



- **Continuous mixing cut**

  The length of arc $(2,3)$ is $\frac{x^2_1 - 69}{98} + \frac{(x^2_2 + x^2_3)}{98} + (0 - \frac{45}{98} + 1)(z^2 - 1) - \frac{45}{98}$.

  The length of arc $(3,2)$ is $\frac{(x^3_2 + x^3_3)}{98} + (\frac{45}{98} - 0)z^3$.

  The continuous mixing inequality is

$$x^2_1 + x^2_2 + x^2_3 + 53z^2 + x^3_2 + x^3_3 + 45z^3 \ge 167. \quad (3)$$

  - If $z^2 = 0$, or $z^2 = 1$ and $z^3 = 0$, trivially satisfied.
  - If $z^2 = z^3 = 1 \Rightarrow x^2_1 + x^2_2 + x^2_3 + x^3_2 + x^3_3 \ge x^2_1 \ge 69$.
- **Cardinality-constrained continuous mixing cut**

  Constraint (2) can be relaxed to $z^2 + z^3 \le 1$. Let $p = 1$, $T = \{2\}$, $H = \{3\}$, the cardinality-constrained continuous mixing inequality is

$$x^2_1 + x^2_2 + x^2_3 + 53z^2 \ge 167,$$

  which is stronger than the continuous mixing inequality (3).
  - If $z^2 = 0 \Rightarrow$ trivially satisfied. If $z^2 = 1 \Rightarrow x^2_1 + x^2_2 + x^2_3 \ge 114$.

Extended Formulations

- **Based on extreme rays and points**

  Let $K := \left|\cup_{T:0 \le |T| \le k} \{P^T_i\}^n_{i=1}\right| = n\left[\binom{n}{k} + \binom{n}{k-1} + \ldots + \binom{n}{1} + \binom{n}{0}\right]$. Denote $\cup_{T:0 \le |T| \le k} \{P^T_i\}^n_{i=1} = \{(\bar{s}^i, \bar{r}^i, \bar{z}^i) : i = 1, \dots, K\}$.

$$\begin{aligned} \mathrm{conv}(Q^k_n) = \mathrm{proj}_{s,r,z} \Big\{ &(s,r,z,\lambda,\mu) \in \mathbb{R} \times \mathbb{R}^n_+ \times [0,1]^n \times \mathbb{R}^K_+ \times \mathbb{R}^{n+2}_+ : \\ &(s,r,z) = \sum_{i=1}^K \lambda_i (\bar{s}^i, \bar{r}^i, \bar{z}^i) + \sum_{j=1}^n \mu_j (0, e_j, \underline{0}) + \mu_{n+1}(1, \underline{0}, \underline{0}) \\ &+ \mu_{n+2}(-1, \underline{1}, \underline{0}), \sum_{i=1}^K \lambda_i = 1, s + r_i + z_i \ge f_i, i = 1, \dots, n \Big\}. \end{aligned}$$

- **Based on disjunctive programming**

$$Q^{1n}_k(f_i - 1) := \{(s,r,z) \in Q^n_k : s = f_i - 1\}$$

$$Q^{2n}_k(f_i) := \{(s,r,z) \in Q^n_k : s = f_i\}$$

  As $s \in \{f_i - 1, f_i\}^n_{i=1}$ in extreme points of $\mathrm{conv}(Q^n_k)$, we have

$$\mathrm{conv}(Q^n_k) = \mathrm{conv}\left(\left(\cup^n_{i=1} \mathrm{conv}(Q^{1n}_k(f_i - 1))\right) \cup \left(\cup^n_{i=1} \mathrm{conv}(Q^{2n}_k(f_i))\right)\right) + C,$$

  where

$$\begin{aligned} C = &\{(s,r,z) \in \mathbb{R} \times \mathbb{R}^n_+ \times \{0,1\}^n : s \ge 0, r = z = \mathbf{0}\} \\ &\cup \{(s,r,z) \in \mathbb{R} \times \mathbb{R}^n_+ \times \{0,1\}^n : r_j \ge 0, s = z = \mathbf{0}\}^n_{j=1} \\ &\cup \{(s,r,z) \in \mathbb{R} \times \mathbb{R}^n_+ \times \{0,1\}^n : s + r_j = 0, j \in [1,n], z = \mathbf{0}\} \end{aligned}$$

  is the recession cone of the linear programming relaxation of Q^n_k.

  From Balas(1998), we have an extended formulation of size $O(n^2) \times O(n^2)$ variables and constraints.

Computations

- **Given**:
  - $\xi_t, \mu_t, \gamma_t, \nu_t$: random variables for cumulative demand, variable and fixed order costs, holding cost in period t, $t \in [1.n]$.
  - $\Gamma_t := (\xi_1, \dots, \xi_t, \mu_1, \dots, \mu_t, \gamma_1, \dots, \gamma_t, \nu_1, \dots, \nu_t)$.
- **Variables**:
  - $x_t(\Gamma_{t-1})$: order quantity in period t, $t \in [1,n]$; $x_1(\Gamma_0) \equiv x_1$.
  - $w_t(\Gamma_{t-1})$: setup variable in period t, $t \in [1,n]$; $w_1(\Gamma_0) \equiv w_1$.
  - $s_t(\Gamma_t)$: inventory at the end of period t, $t \in [1,n]$.
- **Objective**:
  - Minimize the expected total cost.
- **Dynamic probabilistic lot sizing (DPLS) model**:

$$\begin{aligned} \min \ & \mathbb{E}_\Gamma \textstyle\sum_{t=1}^n (\mu_t x_t(\Gamma_{t-1}) + \gamma_t w_t(\Gamma_{t-1}) + \nu_t s_t(\Gamma_t)) \\ \text{s.t.} \ & \mathbb{P} \begin{pmatrix} x_1 & \ge \xi_1 \\ x_1 + x_2(\Gamma_1) & \ge \xi_2 \\ \vdots & \vdots \ \vdots \\ x_1 + x_2(\Gamma_1) + \cdots + x_n(\Gamma_{n-1}) & \ge \xi_n \end{pmatrix} \ge \tau, \\ & x_t(\Gamma_{t-1}) \le M_t w_t(\Gamma_{t-1}) && t \in [1,n], \\ & s_n(\Gamma_n) = \textstyle\sum_{j=1}^n x_j(\Gamma_{j-1}) - \xi_n, \\ & s_t(\Gamma_t) \ge \textstyle\sum_{j=1}^t x_j(\Gamma_{j-1}) - \xi_t && t \in [1, n-1], \\ & s_t(\Gamma_t) \ge 0 && t \in [1,n], \\ & w_t(\Gamma_{t-1}) \in \{0,1\} && t \in [1,n], \\ & x_t(\Gamma_{t-1}) \ge 0 && t \in [1,n]. \end{aligned}$$

- **Methods tested**:
  1. Branch-and-cut with CPLEX cuts,
  2. Branch-and-cut with CPLEX cuts and mixing cuts,
  3. Branch-and-cut with CPLEX cuts, mixing cuts, continuous mixing cuts.
- **Solver**: IBM ILOG CPLEX 12.0

Computations

- Scenario tree: full binary tree
- θ: ratio of fixed and variable costs
- Average of five instances is reported for each setting

$n.m.\tau.\theta$	Alg.	Cuts: CPLEX	MIX	CMIX	RGap	Node	Time(unslvd)	EGap
6.64.85.100	1	3651.0	0.0	0.0	4.68%	11370.8	115.79	0.00%
	2	3616.6	34.4	0.0	4.48%	4462.4	60.80	0.00%
	3	3488.6	40.0	122.4	3.85%	2740.6	31.27	0.00%
6.64.75.150	1	3651.0	0.0	0.0	4.41%	9710.2	100.93	0.00%
	2	3611.2	39.8	0.0	3.87%	5658.6	57.07	0.00%
	3	3473.8	31.2	146.0	3.77%	3684.8	43.53	0.00%
7.128.90.100	1	8451.0	0.0	0.0	10.25%	32125.8	≥3600	3.51%
	2	8387.0	64.0	0.0	9.47%	74472.4	≥3600	2.07%
	3	8222.6	64.4	164.0	8.73%	62909.4	≥3600	1.60%
7.128.90.150	1	8451.0	0.0	0.0	6.89%	67935.2	≥3600	1.54%
	2	8389.8	61.2	0.0	6.15%	81732.2	2970.78(4)	1.16%
	3	8193.8	68.6	188.6	5.73%	44211.6	894.63(3)	0.80%
7.128.85.150	1	8451.0	0.0	0.0	8.57%	92526.4	≥3600	2.88%
	2	8372.6	78.4	0.0	7.52%	61291.8	≥3600	1.52%
	3	8071.6	82.8	296.6	6.87%	49393.2	3115.83(3)	1.14%

Computations

- **Ongoing work**:
  - More valid inequalities for CCMIX in explicit form.
  - Separation algorithm for valid inequalities of CCMIX.
  - Branch-and-cut algorithm with valid inequalities for CCMIX, and computations.
- **References**:

  - Zhang, M., Goel, S., and Küçükyavuz, S. (2011). A branch-and-cut method for dynamic decision-making under joint chance constraints, submitted.
  - Zhang, M. and Küçükyavuz, S. (2012). Cardinality-constrained continuous mixing set, in preparation.