Computing Bounds and Solutions to the Asymmetric Traveling Salesman Problem with Approximate Linear Programming

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Background

Problem Statement

- We consider the asymmetric *traveling salesman problem* (TSP) over cities in $N \cup 0$, where the cost from city i to city j is $c_{i,j} \in \mathbb{R}$ for $i, j = 0, \ldots, |N|$ with $i \neq j$.
- In previous work[(1)], the second author:
  - applied *approximate linear programming* (ALP) techniques to the TSP's *dynamic program* (DP) formulation
  - found a nested family of polyhedral lower bounds, solvable in polynomial time
- In this poster, we present:
  - two methods for solving these ALP formulations to compute TSP lower bounds
  - a branch-and-bound heuristic which uses solutions to the ALP to generate TSP tours via a *price-directed* policy

Lower Bound Framework

Dual of the Dynamic Programming Formulation:

$$\begin{aligned} \max\ & y_{0,N} && \text{(1a)}\\ \text{s.t. } & y_{0,N} - y_{i,N\setminus i} \leq c_{0i}, \quad \forall\, i \in N && \text{(1b)}\\ & y_{i,U\cup j} - y_{j,U} \leq c_{ij}, \quad \forall\, i \in N,\ j \in N \setminus i,\ U \subseteq N \setminus \{i,j\} && \text{(1c)}\\ & y_{i,\emptyset} \leq c_{i0}, \quad \forall\, i \in N && \text{(1d)}\\ & y_{0,N} \in \mathbb{R}; \quad y_{i,U} \in \mathbb{R}, \forall\, i \in N, U \subseteq N \setminus i. && \text{(1e)} \end{aligned}$$

Approximate Cost-to-Go:

$$y_{i,U} = \pi_{i,\emptyset} + \sum_{k\in U} \pi_{i,k} + \sum_{\substack{W\subseteq U\\ |W|\geq n-t}} \lambda_{i,W} + \sum_{\substack{W\subseteq N\setminus(U\cup i)\\ |W|\geq n-t}} \mu_{i,W} \quad \text{for } t \geq 0 \qquad (2)$$

Approximate Linear Programming Formulation:

$$\begin{aligned} \max\ & y_{0,N} && \text{(3a)}\\ \text{s.t. } & y_{0,N} - \pi_{i,\emptyset} - \sum_{k\in N\setminus i} \pi_{i,k} \leq c_{0i}, \quad \forall\, i \in N && \text{(3b)}\\ & \pi_{i,\emptyset} - \pi_{j,\emptyset} + \pi_{i,j} + \sum_{k\in U} (\pi_{i,k} - \pi_{j,k}) \leq c_{ij}, \\ & \qquad \forall\, i \in N, j \in N \setminus i,\ U \subseteq N \setminus \{i,j\} && \text{(3c)}\\ & \pi_{i,\emptyset} \leq c_{i0}, \quad \forall\, i \in N && \text{(3d)}\\ & y_{0,N} \in \mathbb{R}; \quad \pi_{i,\emptyset} \in \mathbb{R}\ \forall\, i \in N,\ \pi_{i,j} \in \mathbb{R}\ \forall\, i \in N, j \in N \setminus i && \text{(3e)} \end{aligned}$$

Note: The above formulation is for the case where $t = 0$. It becomes slightly more complex if $t > 0$. This poster encompasses results where $t = 0$ or $t = 1$.

References

(1) Toriello, A. Optimal Toll Design: A Lower Bound Framework for the Asymmetric Traveling Salesman Problem. Preprint available at $www - bcf.usc.edu/\ toriello/tsp\_bound.pdf$, 2012.

Computing Bounds

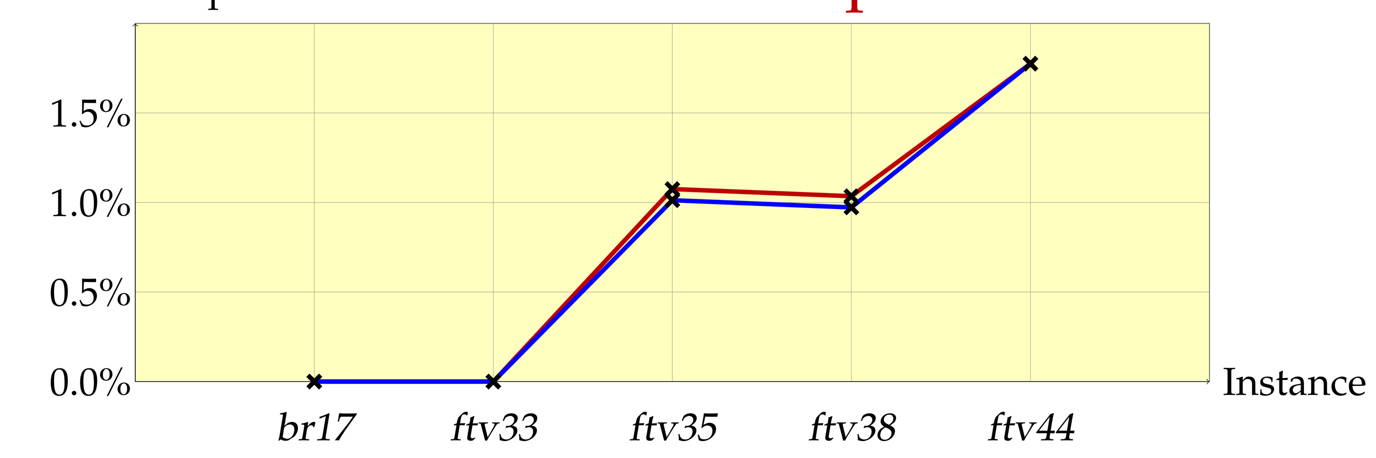
Constraint Generation Algorithm

Method:
This approach solves the formulation directly, using a constraint generation procedure to manage the exponential number of constraints.

Computational Challenges:
In previous work of the second author, he outlines a polynomial-time separation routine for our formulation. Computationally, the main challenge came down to managing the number of constraints to add/remove each iteration.

Results:

Lower Bound Comparison



RED PLOT = bound for the linear relaxation of the arc-based formulation of the TSP
BLUE PLOT = bound for our ALP where t = 1

Note: For the *ftv35* and *ftv38* instances, the bound improvement is roughly 0.06%.

Primal-Dual Algorithm

Method:
This method takes any given feasible solution and attempts to improve it by solving an auxiliary problem in polynomial time to determine a potential direction of improvement.

Computational Challenges:
In order to identify tight constraints in polynomial time, we perform the following for each $i \in N,\ j \in N\setminus i,$ and $k \in N\setminus\{i,j\}$:

1. If $\pi_{i,k} - \pi_{j,k} > 0$, then $k \in U^{+}_{i,j}$.
2. If $\pi_{i,k} - \pi_{j,k} < 0$, then $k \in U^{-}_{i,j}$.
3. If $\pi_{i,k} - \pi_{j,k} = 0$, then $k \in U^{=}_{i,j}$.

To construct the direction LP, include the following class of constraints:

$$\hat{\pi}_{i,\emptyset} - \hat{\pi}_{j,\emptyset} + \hat{\pi}_{i,j} + \textstyle\sum_{k\in U^{+}_{i,j}} (\hat{\pi}_{i,k} - \hat{\pi}_{j,k}) + \sum_{k\in W} (\hat{\pi}_{i,k} - \hat{\pi}_{j,k}) \leq 0,$$
$$\forall (i,j) \in A,\ W \subseteq U^{=}_{i,j},\ t - |U^{+}_{i,j}| \leq |W| \leq (n-t-2) - |U^{+}_{i,j}|$$

Note: If $|U^{+}_{i,j}| > n - t - 2$ or $|U^{+}_{i,j}| + |U^{=}_{i,j}| < t$, a minor change to $U^{+}_{i,j}$ and $U^{=}_{i,j}$ is needed.

Results:
We are currently working to obtain useful results utilizing this Primal-Dual Algorithm for instances of moderate size. Rounding errors have made it difficult to define appropriate tolerances for "tight" constraints.

Generating Tours

Price-Directed Branch-and-Bound Heuristic

Method:
The lower bound solutions can be used to generate TSP tours in a price-directed tour generation heuristic. The approximate costs-to-go derived from a dual solution serve as proxies for the actual costs as shown below:

If we are currently at city i and have cities U left to visit,
the approx. cost of next visiting city $j \in U$ is given by $\{\ c_{i,j} + y_{j,U\setminus j}\ \}$

Computational Challenges:
It is when two cities' approximate costs-to-go are tied that challenges arise. To help us make a decision in this event, we perform further analysis as follows:

1. For each candidate city $k \in \hat{U}$, solve a subproblem defined as the linear relaxation of the arc-based TSP formulation on the remaining cities, U.
   Note: Costs into city k are replaced by costs into the start city ($c_{i,0} \Rightarrow c_{i,k}$).
2. Update the approx. cost of next visiting each city $k \in \hat{U}$.
3. Choose to move to the city of minimum estimated cost.
4. Save the solution to the subproblem in order to calculate relevant costs-to-go later on.
5. In the event that two or more candidate cities are still tied, branch on each tied city.

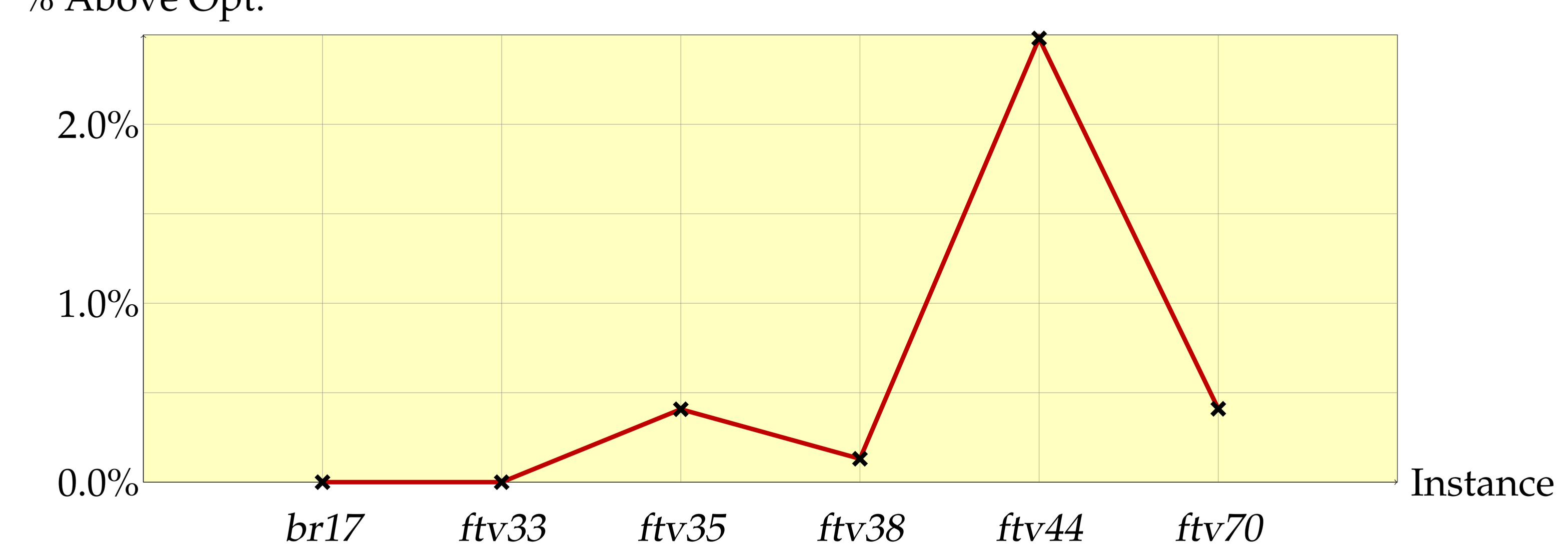
Note: We have experimented with performing Step (1) both more and less frequently.

To reduce the number of iterations, we fathom branches of the tree when:

- The bound on the cost of a partial tour exceeds the cost of a previously-found complete tour (eliminating the need to continue exploring this branch of the tree).
- A min cost subproblem yields an integral optimal solution (immediately giving us a min cost way to complete that partial tour).

Results:

Min Cost Tours Found



RED PLOT = min tour cost we found for the given test instance.

Note: Generally, we are able to find a tour of cost no more than 1-2% greater than the optimal tour cost.

Our heuristic terminated in under 10 min for all instances under 50 cities in size. Results for the 70 city instance took approximately 6 hours to obtain. We are currently running experiments on test instances of over 150 cities.