Introduction

We focus on generation of deep cutting planes through the Gomory–Johnson infinite group relaxation:

\[ x = f + \sum_{r \in \mathbb{R}^k} r s_r \in \mathbb{R}^k \tag{IR} \]

\[ s_r \in \mathbb{Z}_+ \text{ for all } r \in \mathbb{R}^k \]

A valid function is:

- Minimal if there is no valid function \( \hat{\pi} \neq \pi \) such that \( \hat{\pi}(r) \leq \pi(r) \) for all \( r \in \mathbb{R}^k \)
- Extreme if it cannot be written as a convex combination of two other valid functions, i.e., \( \pi = \frac{1}{2} \pi_1 + \frac{1}{2} \pi_2 \) implies \( \pi = \pi_1 = \pi_2 \)
- A Facet if for every valid function \( \pi \), we have that \( S(\pi) \subseteq S(\hat{\pi}) \) implies \( \hat{\pi} = \pi \), where \( S(\pi) \) is the set of all \( r \) satisfying (IR) such that \( \sum_{c \in \mathbb{R}^k} \pi(c) r_s = 1 \).
- Facet \( \Rightarrow \) Extreme \( \Rightarrow \) Minimal.

Valid functions. We start by defining the analog of a cut for the infinite relaxation. We say that a function \( \pi : \mathbb{R}^k \to \mathbb{R} \) is valid for (IR) if \( \pi \geq 0 \) and the inequality

\[ \sum_{r \in \mathbb{R}^k} \pi(r) s_r \geq 1 \]  

is satisfied by every feasible solution \( s \) of (IR).

The following theorem was generalized to 2-dimensions by Cornuésjols and Molinaro. We provide a generalization to \( k \)-dimensions. Theorem (Gomory–Johnson 2-Slope Theorem) Let \( \pi : \mathbb{R} \to \mathbb{R} \) be a minimal valid function. If \( \pi \) is a continuous piecewise linear function with only two slopes, then \( \pi \) is a facet.

Definition A function \( \theta : \mathbb{R}^k \to \mathbb{R} \) is genuinely \( k \)-dimensional if there does not exist a function \( \varphi : \mathbb{R}^{k-1} \to \mathbb{R} \) and a linear map \( T : \mathbb{R}^k \to \mathbb{R}^{k-1} \) such that \( \varphi = \theta \circ T \).

Theorem Let \( \pi : \mathbb{R}^k \to \mathbb{R} \) be a minimal valid function that is piecewise linear with a locally finite cell complex and genuinely \( k \)-dimensional with at most \( k + 1 \) slopes. Then \( \pi \) is a facet.

Proof of Theorem

Let \( \pi \) have a locally finite, complete polyhedral complex \( P \) in \( \mathbb{R}^k \) and let \( \{ P_i \}_{i=1} \) be a partition of the set of maximal cells of \( P \) where \( \pi \) has a distinct slope \( g_i \) in each \( P_i \). We consider any minimal valid function \( \bar{\pi} \) such that \( E(\bar{\pi}) \subseteq E(\pi) \) and that \( \bar{\pi} = \pi \).

Proposition The function \( \bar{\pi} \) is a piecewise linear function compatible with \( \{ P_i \}_{i=1} \).

Proof. Fix \( i \in \{ 1, \ldots, k + 1 \} \). There exists a maximal cell \( P_i \) in \( P \), containing the origin. Since \( P_0 \) is a full-dimensional polyhedron containing the origin, there exists a full-dimensional parallelotope \( \Pi \subseteq \mathbb{R}^k \) such that \( \bar{\pi}(r) = g_i \cdot r \) for \( r \in \Pi \). Now let \( P \) be any maximal cell in \( P_i \) and pick any \( y \in \text{relint}(P) \). By applying the interval lemma to translates of \( P_{i-1} \), we show \( \bar{\pi} \) is a piecewise linear function compatible with \( \{ P_i \}_{i=1} \). \( \square \)

Proposition \( \pi \) and \( \bar{\pi} \) are both Lipschitz continuous.

Constructing a system of linear equations

We construct a system of linear equations which is satisfied by both \( g_i, g_i^{+1}, g_i^{-1} \), and \( g_i, g_i^{+1}, g_i^{-1} \).

Lemma \( \exists \) vectors \( r, r', r'' \in \mathbb{R}^k \) such that the system has a unique solution.

(i) For every \( i, j, \ell \in \{ 1, \ldots, k + 1 \} \) with \( i, j, \ell \) different from \( i \), the equations \( r \cdot g_i = r' \cdot g_i' = r'' \cdot g_i'' \) hold.

(ii) \( \sum_{i=1}^{k+1} (\pi_i(a_i)) \cdot g_i = 1 \) for all \( a_i \in \mathbb{R}^{k+1} \).

Proof. Consider the neighborhood \( B_i(0) \). Let \( F_i = \bigcup_{P \in \mathcal{P}(P)} (P \cap B_i(0)) \).

The set \( F_i \) is disjoint with \( H_i \) for \( i = 1, \ldots, k \).

For \( I \subseteq \{ 1, \ldots, k + 1 \} \), let \( C_I = \bigcap_{i \in I} H_i \). It follows that for all \( i \notin I \), \( F_i \) disjoint with \( C_I \). Alternately, the gradient of \( \pi \) in any point in \( C_I \cap B_i(0) \) must be within the set \( \{ g_i^{1}, g_i^{-1} \} \). \( \square \)

System of Equations

Corollary Choose \( a_1, a_2, \ldots, a^{k+1} \in \mathbb{R}^k \) such that \( \text{cone}(a_1, a_2, \ldots, a^{k+1}) = \mathbb{R}^k \).

Proof. Since the system has a unique solution, \( g_i = g_i' = g_i'' \) for \( i = 1, \ldots, k \), and hence \( \pi = \bar{\pi} \). \( \square \)