A $(k+1)$-Slope Theorem for the k-Dimensional Infinite Group Relaxation

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Introduction

We focus on generation of deep cutting planes through the Gomory–Johnson *infinite group relaxation* :

$$x = f + \sum_{r \in \mathbb{R}^k} r s_r \in \mathbb{Z}^k \qquad \text{(IR)}$$
$$s_r \in \mathbb{Z}_+ \quad \text{for all } r \in \mathbb{R}^k$$
$$s \text{ has finite support.}$$

Valid functions. We start by defining the analog of a cut for the infinite relaxation. We say that a function $\pi\colon \mathbb{R}^k \to \mathbb{R}$ is *valid* for (IR) if $\pi \geq 0$ and the inequality

$$\sum_{r \in \mathbb{R}^k} \pi(r) s_r \geq 1 \qquad (1)$$

is satisfied by every feasible solution s of (IR).

A valid function is:

- *Minimal* if there is no valid function $\tilde\pi \neq \pi$ such that $\tilde\pi(r) \leq \pi(r)$ for all $r \in \mathbb{R}^k$,
- *Extreme* if it cannot be written as a convex combination of two other valid functions, i.e., $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$ implies $\pi = \pi_1 = \pi_2$.
- *A Facet* if for every valid function $\tilde\pi$, we have that $S(\pi) \subseteq S(\tilde\pi)$ implies $\tilde\pi = \pi$, where $S(\pi)$ is the set of all s satisfying (IR) such that $\sum_{r\in\mathbb{R}^k} \pi(r) s_r = 1$.

Facet $\Rightarrow$ Extreme $\Rightarrow$ Minimal

The following theorem was generalized to 2-dimensions by Cornuéjols and Molinaro. We provide a generalization to k-dimensions.

Theorem[Gomory–Johnson 2-Slope Theorem] Let $\pi\colon \mathbb{R} \to \mathbb{R}$ be a minimal valid function. If π is a continuous piecewise linear function with only two slopes, then π is a facet.

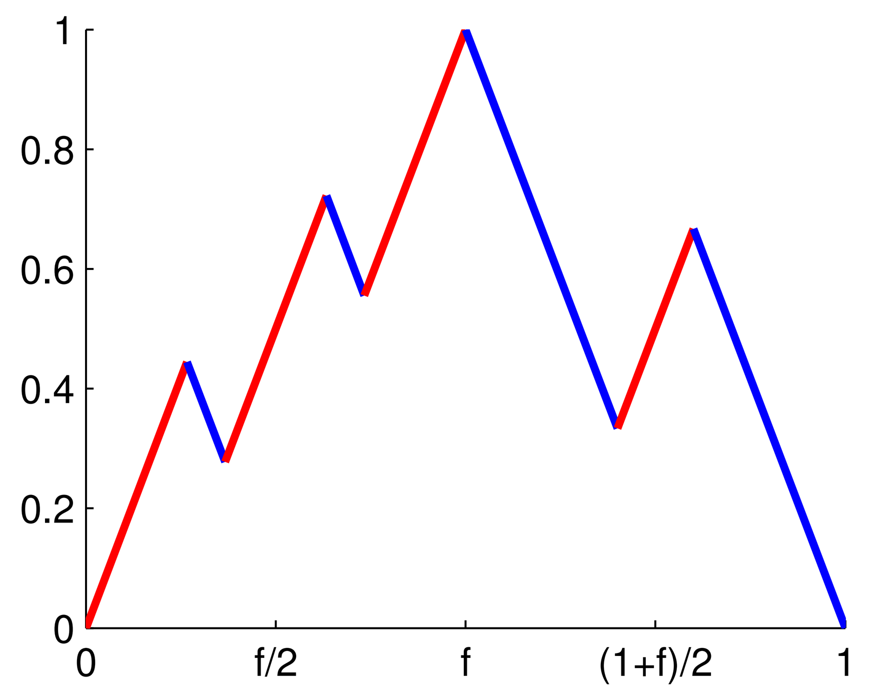
Definition A function $\theta\colon \mathbb{R}^k \to \mathbb{R}$ is *genuinely k-dimensional* if there does not exist a function $\varphi\colon \mathbb{R}^{k-1} \to \mathbb{R}$ and a linear map $T\colon \mathbb{R}^k \to \mathbb{R}^{k-1}$ such that $\theta = \varphi \circ T$.

Theorem

Let $\pi\colon \mathbb{R}^k \to \mathbb{R}$ be a minimal valid function that is piecewise linear with a locally finite cell complex and genuinely k-dimensional with at most $k+1$ slopes. Then π is a facet.

Preliminaries

- **Theorem** [Minimality Theorem, GJ] Let $\pi\colon \mathbb{R}^k \to \mathbb{R}$ be a non-negative function. Then π is a minimal valid function for (IR) if and only if $\pi(0) = 0$, π is periodic with respect to $\mathbb{Z}^k$, subadditive, i.e., $\pi(a+b) \leq \pi(a) + \pi(b)$ for all $a, b \in \mathbb{R}^k$, and satisfies the symmetry condition, i.e., $\pi(r) + \pi(-f-r) = 1$ for all $r \in \mathbb{R}^k$.
- **Theorem** [Facet Theorem] Let π be a minimal valid function. Suppose for every minimal valid function $\tilde\pi$, we have that $E(\pi) \subseteq E(\tilde\pi)$ implies $\tilde\pi = \pi$, where $E(\theta)$ denotes the set of all pairs $(u, v) \in \mathbb{R}^k \times \mathbb{R}^k$ such that $\theta(u+v) = \theta(u) + \theta(v)$. Then π is a facet.
- **Lemma** [Interval Lemma, GJ] Let $\theta\colon \mathbb{R} \to \mathbb{R}$ be a function bounded on every bounded interval. Given real numbers $u_1 < u_2$ and $v_1 < v_2$, let $U = [u_1, u_2]$, $V = [v_1, v_2]$, and $U + V = [u_1 + v_1, u_2 + v_2]$. If $\theta(u) + \theta(v) = \theta(u+v)$ for every $u \in U$ and $v \in V$, then there exists $c \in \mathbb{R}$ such that
$$\begin{aligned} \theta(u) &= \theta(u_1) + c(u - u_1) && \text{for every } u \in U, \\ \theta(v) &= \theta(v_1) + c(v - v_1) && \text{for every } v \in V, \\ \theta(w) &= \theta(u_1 + v_1) + c(w - u_1 - v_1) && \text{for every } w \in U + V. \end{aligned}$$
- **Lemma** [KKM] Consider an n-simplex $\operatorname{conv}(u^j)_{j=1}^n$. Let $F_1, F_2, \dots, F_n$ be closed sets such that for all $I \subseteq \{1, \dots, n\}$, the face $\operatorname{conv}(u^j)_{j\in I}$ is contained in $\bigcup_{j\in I} F_j$. Then the intersection $\bigcap_{j=1}^n F_j$ is non-empty.



Proof of Theorem

Let π have a locally finite, complete polyhedral complex $\mathcal{P}$ in $\mathbb{R}^k$ and let $\{\mathcal{P}_i\}_{i=1}^{k+1}$ be a partition of the set of maximal cells of $\mathcal{P}$ where π has a distinct slope g^i in each $\mathcal{P}_i$. We consider any minimal valid function $\tilde\pi$ such that $E(\pi) \subseteq E(\tilde\pi)$ and show that $\tilde\pi = \pi$.

Proposition The function $\tilde\pi$ is a piecewise linear function compatible with $\{\mathcal{P}_i\}_{i=1}^{k+1}$.

Proof. Fix $i \in \{1, \dots, k+1\}$. There exists a maximal cell $P_0 \in \mathcal{P}_i$ containing the origin. Since P_0 is a full-dimensional polyhedron containing the origin, there exists a full-dimensional parallelotope Π with $0 \in \Pi$ and $\Pi + \Pi \subseteq P_0$. Using the interval lemma, there exists a g'_i such that $\tilde\pi(r) = g'_i \cdot r$ for $r \in \Pi$. Now let P be any maximal cell in $\mathcal{P}_i$ and pick any $y \in \operatorname{relint}(P)$. By applying the interval lemma to translates of P_0, we show $\tilde\pi$ is a piecewise linear function compatible with $\{\mathcal{P}_i\}_{i=1}^{k+1}$. □

Proposition $\pi, \tilde\pi$ are both Lipschitz continuous.

Constructing a system of linear equations

We construct a system of linear equations which is satisfied by both $g^1, \dots, g^{k+1}$ and $\tilde g^1, \dots, \tilde g^{k+1}$.

Lemma $\exists$ vectors $r^1, r^2, \dots, r^{k+1} \in \mathbb{R}^k$ with:

(i) For every $i, j, \ell \in \{1, \dots, k+1\}$ with j, ℓ different from i, the equations $r^i \cdot \bar g^j = r^i \cdot \bar g^\ell$ and $r^i \cdot \tilde g^j = r^i \cdot \tilde g^\ell$ hold.

(ii) $\operatorname{cone}(r^i)_{i=1}^{k+1} = \mathbb{R}^k$.

Proof. We consider the neighborhood $B_\varepsilon(0)$. Let $F_i = \bigcup_{P\in\mathcal{P}_i} (P \cap \bar B_\varepsilon(0))$.

- The set F_i is disjoint with H_i for $i = 1, \dots, k$.

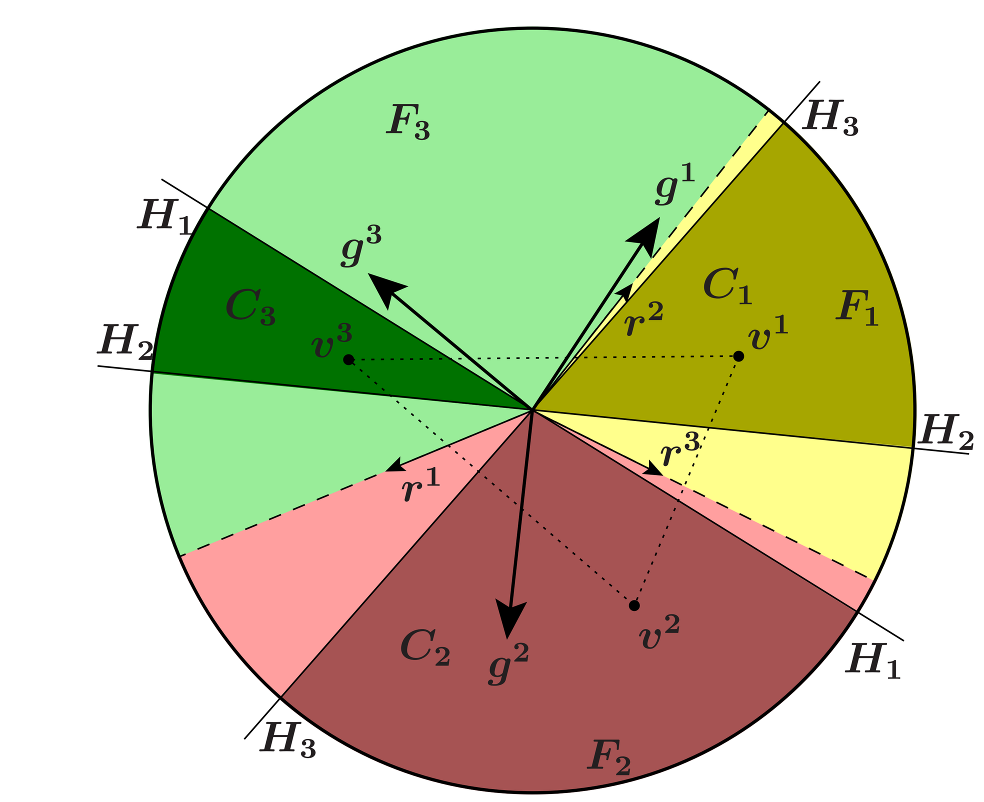
For $I \subseteq \{1, \dots, k+1\}$, let $C_I = \bigcap_{i\notin I} H_i$. It follows that for all $i \notin I$, F_i disjoint with C_I. Alternatively, the gradient of π in any point in $C_I \cap B_\varepsilon(0)$ must be within the set $\{\bar g^i\}_{i\in I}$.

- C_j is full-dimensional for all $j = 1, \dots, k+1$.

Pick $v^j \in \operatorname{int}(C_j) \cap B_\varepsilon(0)$ for $j = 1, \dots, k+1$, then, $v^j \cdot \bar g^i < 0$ for all $i \neq j$ and $\operatorname{cone}(v^i)_{i=1}^{k+1} = \mathbb{R}^k$. $\Delta = \operatorname{conv}(v^i)_{i=1}^{k+1}$ is a full-dimensional simplex. Since $\Delta \subseteq B_\varepsilon(0) \subseteq \bigcup_{i=1,\dots,k+1} F_i$, the sets F_i form a closed cover of Δ. In particular they form a closed cover of each facet $\Delta_i = \operatorname{conv}(v^j)_{j\neq i}$.

- for every $I \subseteq \{1, \dots, k+1\} \setminus \{i\}$, the face $\operatorname{conv}(v^j)_{j\in I}$ is contained in $\bigcup_{j\in I} F_j$.

Therefore, for each $i = 1, \dots, k+1$, the KKM Lemma implies the existence of a point $r^i \in \Delta_i$ belonging to $\bigcap_{j\neq i} F_j$ as desired. Properties (i) and (ii) follow. □



System of Equations

Corollary Choose $a^1, a^2, \dots, a^{k+1} \in \mathbb{Z}^k - f$ such that $\operatorname{cone}(a^i)_{i=1}^{k+1} = \mathbb{R}^k$. Let $r^1, r^2, \dots, r^{k+1}$ be as derived above. Then there exist $\mu_{ij} \in \mathbb{R}_+$, $i, j \in \{1, \dots, k+1\}$ with $\sum_{j=1}^{k+1} \mu_{ij} = 1$ for all $i \in \{1, \dots, k+1\}$ such that both $\tilde g^1, \dots, \tilde g^{k+1}$ and $\bar g^1, \dots, \bar g^{k+1}$ are solutions to the linear system

$$\begin{aligned} \textstyle\sum_{j=1}^{k+1} (\mu_{ij} a^i) \cdot g^j = 1 \qquad & \forall i \in \{1, \dots, k+1\}, \\ r^i \cdot g^j - r^i \cdot g^\ell = 0 \ \forall i \neq j, \ell & \in \{1, \dots, k+1\} \end{aligned}$$

with variables $g^1, \dots, g^{k+1} \in \mathbb{R}^k$. Rewriting the system reveals it is invertible.

$$\left[\begin{array}{cc|ccc|c} 1 & a^1 & \mu_{11} a^1 & \dots & \mu_{1(k+1)} a^1 & O_{1\times k} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & a^{k+1} & \mu_{(k+1)1} a^{k+1} & \dots & \mu_{(k+1)(k+1)} a^{k+1} & O_{1\times k} \\ \hline O_{k\times 1} & O_{k\times k} & R_1 & & & -I_1 \\ \vdots & \vdots & & \ddots & & \vdots \\ O_{k\times 1} & O_{k\times k} & & & R_{k+1} & -I_{k+1} \end{array}\right]$$

Since the system has a unique solution, $g^i = \tilde g^i$ for $i = 1, \dots, k$, and hence $\pi = \tilde\pi$. ■