A Polynomial Procedure For Generating Generalized Intersection Cuts

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Full Hyperplane Activation

Focus: Non recursive cut generation for mixed integer programming (MIP) problems

MIP: $\min\{cx : Ax \geq b, x \geq 0,\ x_j \in \mathbb{Z},\ j \in \mathcal{N}_I \subseteq \mathcal{N} := [0,1,\ldots,N]\}$

Linear relaxation feasible set, P: $\{x : Ax \geq b,\ x \geq 0\}$

Integer hull, P_I: $P \cap \{x : x_j \in \mathbb{Z}, j \in \mathcal{N}_I\}$

Issue: Standard recursive cutting plane procedures suffer from numerical issues caused by dual degeneracy (Balas et al. 2010; Zanette et al. 2010)

Solution paradigm: Non-recursive cut generation using intersection points

1. Generate and store the collection of intersection points created by intersecting edges of a relaxation C of P with the boundary of a convex set S such that $int(S) \cap P_I = \emptyset$
2. Use these intersection points to generate deeper generalized intersection cuts (GICs) without recursion

Full hyperplane activation:

Let: C^1 be the cone defined by the optimal simplex tableau

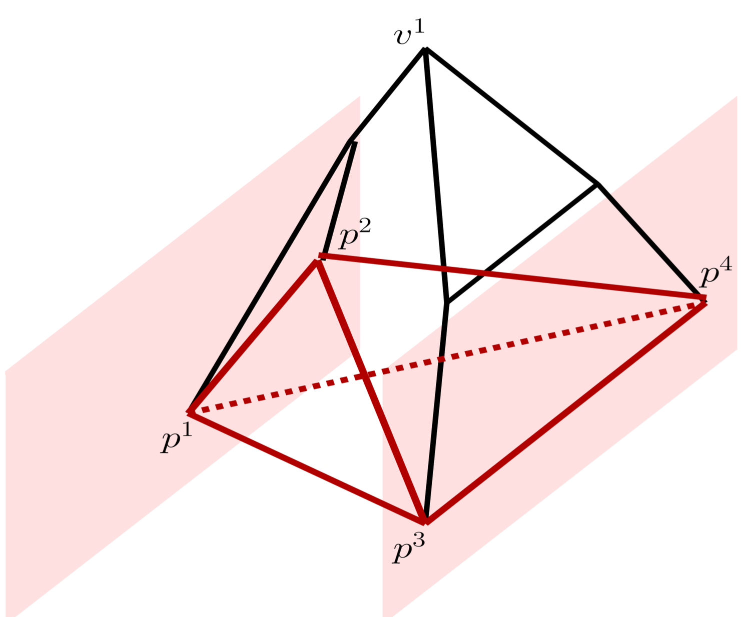
$q_j = r^j \cap \mathrm{bd}S$, where r^j is a ray of C^1

0. Initialize: $\mathcal{P} = \{q^j\}_{j=1}^N$ and $C = C^1$
1. While $n \leq n_h$
   - (a) **Select** halfspace H^+
   - (b) **Remove** from $\mathcal{P}$ all intersection points cut off by H^+
   - (c) **Add** intersection points created by edges of $C \cap H$ with bd$S$
   - (d) **Update** $C = C \cap H^+$
   - (e) $n = n + 1$

Valid cuts: Basic feasible solution $\bar{\alpha}$ to

$$\alpha p^j \geq \bar{\beta},\ \ \forall j \in Q$$

for some $\bar{\beta} \in \{-1, 0, 1\}$ that cuts off a point "above" $\mathrm{conv}(\{p^j\}_{j \in \mathcal{P}})$



Computational issues with full hyperplane activation:

1. Requires a description of the finite and infinite edges of a general polyhedron C
2. The number of edges of C, and hence the number of intersection points, **grow exponentially** with the number of hyperplane activations

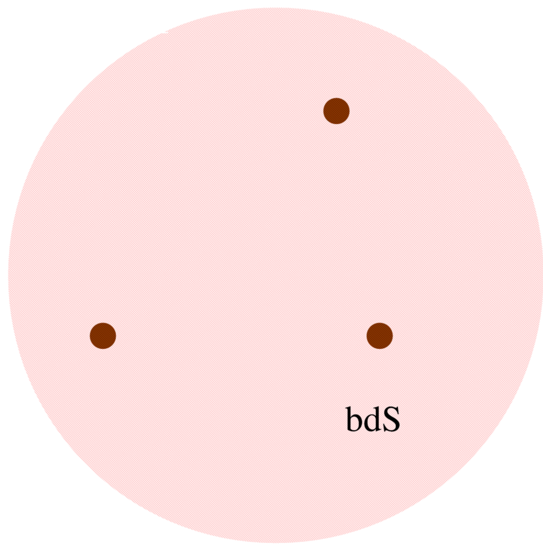
Partial Hyperplane Activation

A streamlined procedure that generates at most $n_h^2 N$ intersection points after n_h partial hyperplane activations

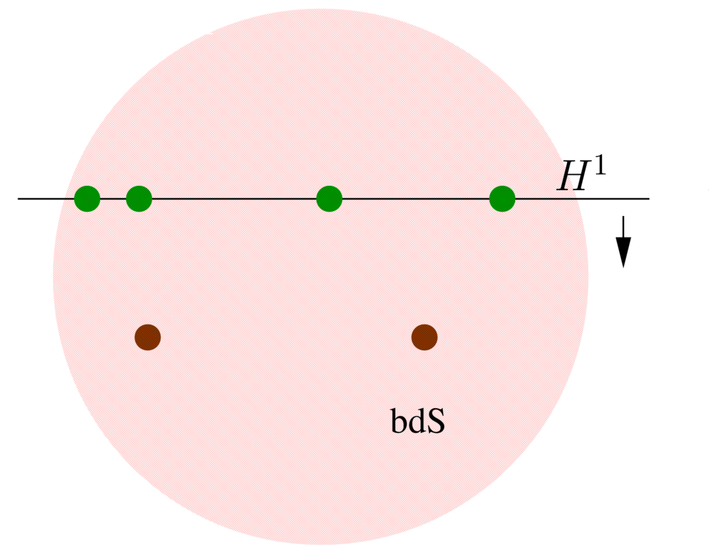
Main ideas:

1. Use a polyhedron C that is a relaxation of P but not defined by the facets of P
2. Activate hyperplanes on C^1 alone: Replace $C \cap H$ in step 1(c) with $C^1 \cap H$ and remove step 1(d)

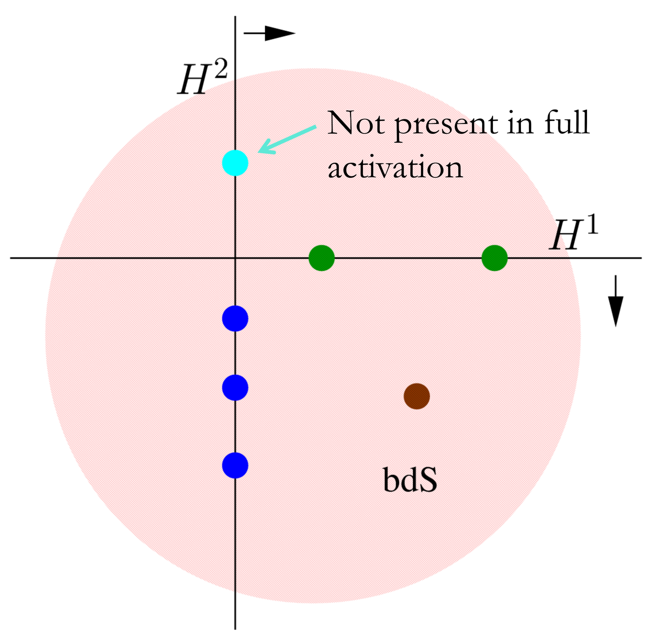
Patch on the bdS with intersection points created by edges of C^1



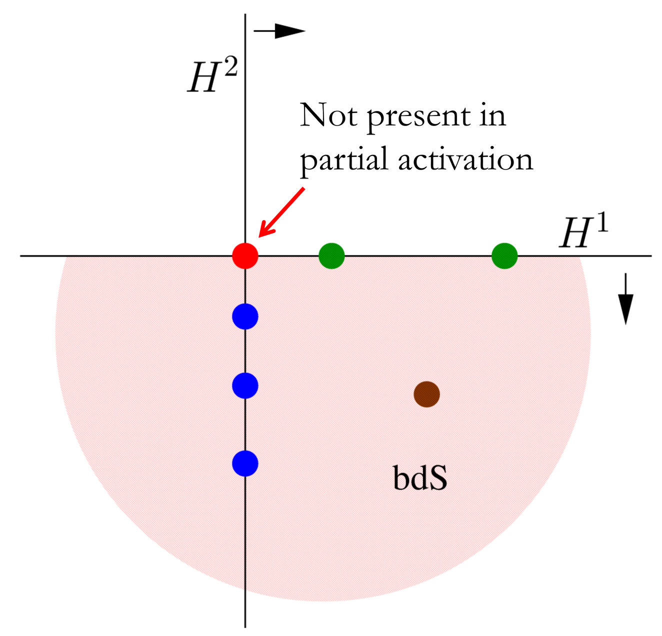
Intersection points after first full or partial hyperplane activation



Intersection points after second partial hyperplane activation



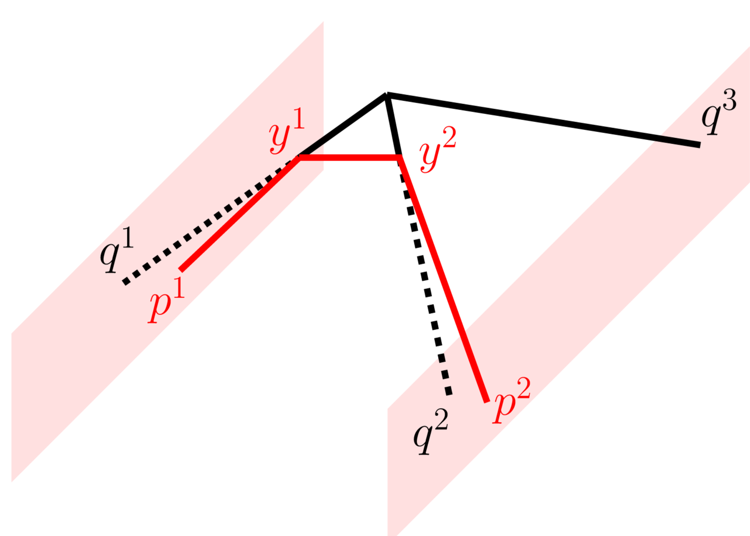
Intersection points after second full hyperplane activation



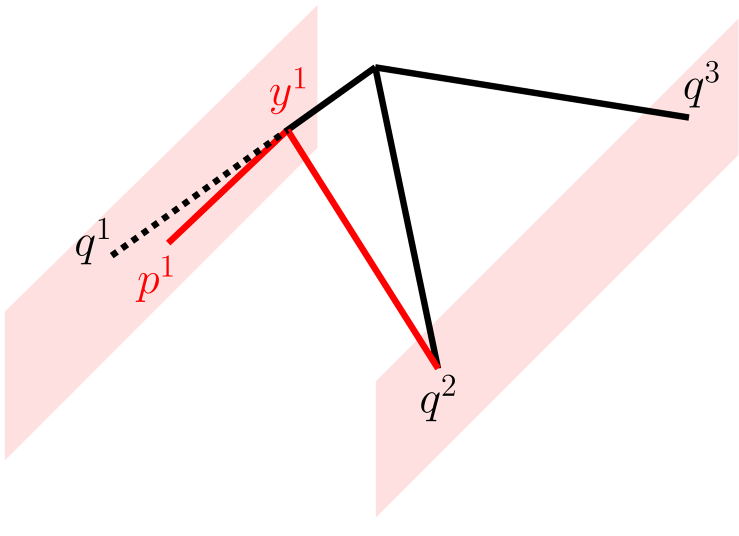
Why "partial"?

When activating a hyperplane H on C^1, we only generate new intersection points from a subset of rays intersected by H

Full activation



Partial activation



Computational Results

- σ denotes the index set of all integer variables fractional in the optimal LP solution
- One round of standard intersection cuts (SICs): $|\sigma|$ SICs, one from each row $x_i, i \in \sigma$
- One round of GICs: $|\sigma|$ GICs generated using simple splits for each $i \in \sigma$
- We compare the gap closed by adding one round of SICs with one round of GICs

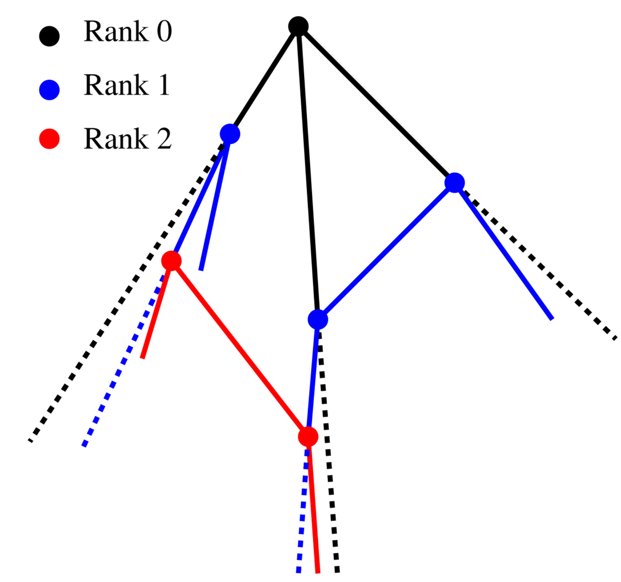
Table 1: Percentage gap closed and CPU time

Instance	$\lvert\sigma\rvert$	n^h	SIC	GIC	GIC − SIC	CPU Time (s)
p0201	13	11	0.00	90.29	90.29	2.95
egout	40	4	51.93	65.09	13.16	0.90
bell3a	32	18	44.74	56.85	12.11	5.11
mod013	5	26	4.40	10.82	6.42	0.11
stein15*	5	3	50.00	54.78	4.78	0.07
bell5	25	13	14.53	18.25	3.72	2.61
bell4	46	13	23.37	25.68	2.31	3.68
bm23	6	3	5.93	7.03	1.10	0.02
misc05	15	17	3.60	4.48	0.87	12.60
lseu	12	5	4.57	5.30	0.73	0.19
p0033	6	7	1.83	2.51	0.68	0.04
bell3b	36	19	44.57	44.91	0.34	6.60
mas74	12	20	3.30	3.55	0.25	19.96
sentoy	8	3	10.38	10.58	0.19	0.13
mas76	11	20	2.37	2.38	0.01	24.42
gt2	11	25	83.13	83.13	0	1.37
sample2	12	25	5.86	5.86	0	1.07

*: objective function $\sum_{i=1}^{15} i x_i$

Conclusions

- Our partial hyperplane activation procedure generates at most $n_h^2 N$ intersection points after n_h partial hyperplane activations
- Computational results show that GICs generated using these points improve on standard intersection cuts in a significant manner
- We are currently testing a substantially faster implementation: O(N) reduction in number of operations required to generate an intersection point.
- **Extension:** A hierarchy of hyperplane activation procedures, where procedures of **higher rank** generate deeper intersection points than ones of lower rank
- **Vertex rank:** Length of shortest path from the apex of C^1
- **Our partial activation procedure is a rank 1 procedure.**
- **Rank 2 procedure:** Repeatedly activate a pair of hyperplanes on C^1



References:

1. Balas, E., M. Fischetti, A. Zanette. 2010. On the enumerative nature of Gomory's dual cutting plane method. Mathematical Programming B, 125(2): 325-351.
2. Balas, E., F. Margot. 2011. Generalized intersection cuts and a new cut generating paradigm. Mathematical Programming A, DOI: 10.1007/s10107-011-0450-6.
3. Zanette, A., M. Fischetti, and E. Balas. 2011. Lexicography and degeneracy: Can a pure cutting plane algorithm work? Mathematical Programming B, 130(1):153-176.