Split Cuts for Conic Programming

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Introduction

We study split cuts for the second-order conic sets of the form

$$LC\,(B,c) := \{(x,t_0) \in \mathbb{R}^n \times \mathbb{R}_+ : \|B(x-c)\| \le t_0\}\,,$$

Where B is an invertible matrix.

In the context of the same set, Atamturk et al. introduced an extended formulation in a higher dimensional space. Introducing auxiliary variables $t \in \mathbb{R}^n_+$, one can reformulate $LC\,(B,c)$ as

$$|b_i(x-c_i)| \le t_i, \quad i = 1,\dots,n$$
$$\|t\| \le t_0,$$

where b_i and c_i denote the ith row of matrix B and vector c, respectively.

Simple conic MIR

Proposition (Simple Conic MIR). *Let*

$$S_0 = \{(x,t) \in \mathbb{Z} \times \mathbb{R}_+ \,:\, |x-b| \le t\},$$

and $f = b - \lfloor b \rfloor$. *Then*

$$(1-2f)(x - \lfloor b \rfloor) + f \le t \tag{1}$$

is valid for S_0 *and* $\mathrm{conv}(S_0) = \{(x,t) \in \mathbb{R} \times \mathbb{R}_+ \,:\, |x-b| \le t, (1)\}$.

Superadditive conic MIR

Theorem (Superadditive Conic MIR). *Let* $S^+ = \{x \in \mathbb{Z}^n_+, t \in \mathbb{R} : |a^Tx - b| \le t\}$, *and let*

$$\phi_f(a) = -a + 2(1-f)\left(\lfloor a \rfloor + \frac{(a - \lfloor a \rfloor - f)^+}{1-f}\right).$$

Then for any $\alpha \ne 0$

$$\sum_{j=1}^{n} \phi_{f_\alpha}(a_j/\alpha)x_j - \phi_{f_\alpha}(b/\alpha) \le \frac{t}{|\alpha|},$$

where $f_\alpha = b/\alpha - \lfloor b/\alpha \rfloor$, *is valid for* S^+.

Conic aggregation: Let

$$P = \{x \in \mathbb{R}^n, y \in \mathbb{R}^p : |Ax - b| \le t\}\,,$$

and $\lambda, \mu \in \mathbb{R}^m_+$. Then

$$\left|\left(\frac{\mu-\lambda}{2}\right)^T t + \left(\frac{\lambda+\mu}{2}\right)^T (Ax-b)\right| \le \left(\frac{\lambda+\mu}{2}\right)^T t + \left(\frac{\mu-\lambda}{2}\right)^T (Ax-b)$$

is a valid inequality for P.

Notations

Definition. *Given a convex set* $C \in \mathbb{R}^n$ *and* $(\pi,\pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$, *define*

$$C^{\pi,\pi_0} := \mathrm{conv}\left((C \cap \{x \in \mathbb{R}^n : \langle \pi, x\rangle \le \pi_0\}) \cup (C \cap \{x \in \mathbb{R}^n : \langle \pi, x\rangle \ge \pi_0 + 1\})\right).$$

Define split cuts of C as any valid (linear or non-linear) inequality for C^{π,π_0} for some $(\pi,\pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$ and the split closure of C as the intersection of all C^{π,π_0} for every possible $(\pi,\pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$.

References

Atamturk, A. and Narayanan, V. (2010), 'Conic mixed-integer rounding cuts', *Mathematical Programming* **122**, 1-20.

Dadush, D., Dey, S., Vielma, J. P. (2011), 'The Split closure of a strictly convex body', *Operations Research Letters* **39**, 121-126.

Vielma, J. P. (2007), 'A constructive characterization of the split closure of a mixed-integer linear program', *Operations Research Letters* **35**, 29-35.

Equivalency of Conic MIR and Split Cuts

Proposition. *Every superadditive conic MIR is a split cut for* S^+.

Moreover, we can show that split cuts and conic MIR cuts are equivalent even in the absence of non-negativity for the integer variables.

Proposition. *Let* $\lambda \in \mathbb{R}^m$ *be such that* $A^T\lambda = \pi \in \mathbb{Z}^n$ *and* $\lambda^T b \notin \mathbb{Z}$. *Then*

$$P^{\pi,\pi_0} := \{(x,t,t_0) \in \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}_+ : \quad |Ax-b| \le t,$$
$$(1-2f)\left(\pi^T x - \lfloor \lambda^T b \rfloor\right) + f \le |\lambda|^T t\},$$

where $\pi_0 = \lfloor \lambda^T b \rfloor$, *and* $f = \lambda^T b - \lfloor \lambda^T b \rfloor$.

This proposition shows that split cuts can be obtained by applying the simple conic MIR to the simple aggregation introduced in the next lemma.

Lemma. *let* $\lambda \in \mathbb{R}^m$. *Then*

$$\left|\lambda^T (Ax-b)\right| \le |\lambda|^T t$$

is a valid inequality for P.

Nonlinear Split Cuts

It is known that when B is the identity matrix and the disjunction is elementary, the split cut of $LC\,(B,c)$ reduces to the conic MIR cut. However, this is not true in general. Fortunately, formulas for nonlinear split cuts can be obtained in the general case.

Proposition. *Let* $B \in \mathbb{R}^{n\times n}$ *be an invertible matrix. Also let* $c \in \mathbb{R}^n$, $(\pi,\pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$ *such that* $\pi_0 < \langle \pi, c\rangle < \pi_0 + 1$. *We have*

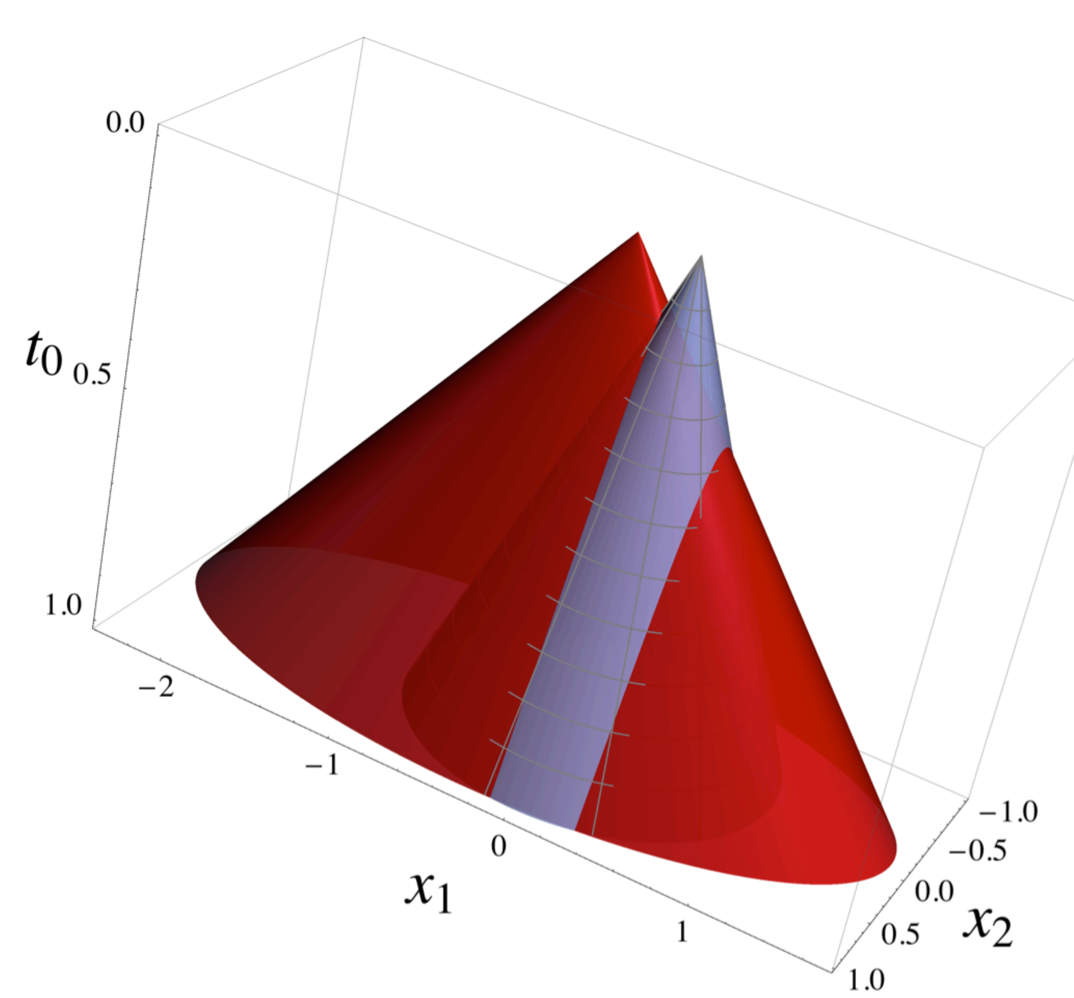
$$LC\,(B,c)^{\pi,\pi_0} := \{(x,t_0) \in \mathbb{R}^n \times \mathbb{R}_+ : \quad \|B\,(x-c)\,\| \le t_0,$$
$$\|A_{B,\pi,c}x - b_{B,\pi,c}\| \le t_0\},$$

where

$$A_{B,\pi,c} = \left(I - \frac{B^{-T}\pi\pi^T B^{-1}}{\|B^{-T}\pi\|^2} + \frac{\pi_1 + \pi_0 - 2\langle \pi, c\rangle}{\pi_1 - \pi_0}\frac{B^{-T}\pi\pi^T B^{-1}}{\|B^{-T}\pi\|^2}\right)B,$$

and

$$b_{B,\pi,c} = \frac{2\,(\pi_1 - \langle \pi, c\rangle)\,(\pi_0 - \langle \pi, c\rangle)\,B^{-T}\pi}{(\pi_1 - \pi_0)\,\|B^{-T}\pi\|^2} + A_{B,\pi,c}Bc.$$



Nonlinear split cuts for ellipsoids

Let

$$E\,(B,c) := \{x \in \mathbb{R} : \|B(x-c)\| \le 1\}\,,$$

and

$$P^{\perp}_{B^{-T}\pi} = I - \frac{B^{-T}\pi\pi^T B^{-1}}{\|B^{-T}\pi\|^2}.$$

Given a split disjunction $(\pi,\pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$, the associated split cut is of the form

$$\|P^{\perp}_{B^{-T}\pi}B\,(x-c)\,\| \le (\pi_0 + 1 - \langle \pi, x\rangle)\sqrt{1 - \left(\frac{\pi_0 - \langle \pi, c\rangle}{\|B^{-T}\pi\|}\right)^2} + (\langle \pi, x\rangle - \pi_0)\sqrt{1 - \left(\frac{\pi_0 + 1 - \langle \pi, c\rangle}{\|B^{-T}\pi\|}\right)^2}.$$

Nonlinear Split Cuts vs. Conic MIR Cuts

A single split disjunction

We can show that given a single split disjunction, the associated nonlinear split cut dominates the conic MIR cut, and such dominance can be strict.

Proposition. *Consider the set* $LC\,(B,c)$ *as defined before. Given a single disjunction* $(\pi,\pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$, *the associated nonlinear split cut dominates conic MIR cut.*

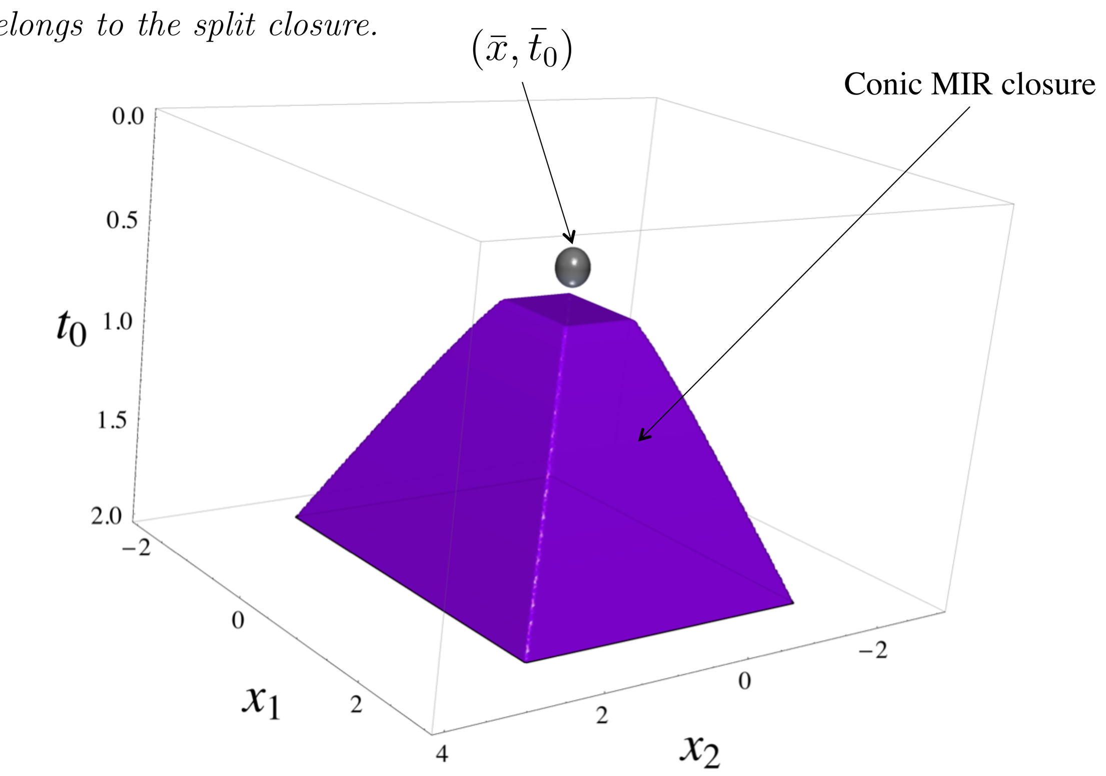
A group of split disjunctions

When there are more than one split disjunctions, it is possible that the conic MIR cuts provide a better, and in some cases arbitrarily better, bound than the nonlinear split cuts.

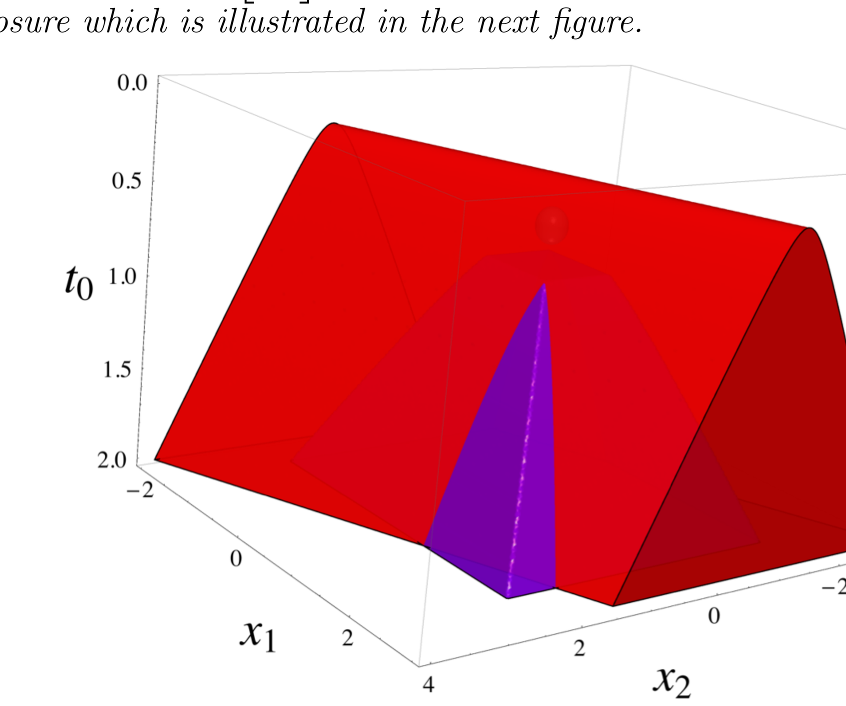
Example. *Consider the second-order cone*

$$LC := \{(x,t_0) \in \mathbb{R}^n \times \mathbb{R}_+ : \|x - c\| \le t_0\}\,,$$

where $c_i = 1/2$ *for* $i \in \{1,\dots,n\}$. *Also consider the elementary disjunctions* $\pi^i = e^i$, *where* e^i *denotes the i-th unit vector. One can show that the elementary conic MIR cuts give the conic MIR closure. Now minimizing* t_0 *over conic MIR closure, one gets the optimal value of* $\sqrt{n}/2$, *which is in fact the optimal value of the IP problem (this is independent of c). However, one can show that the point* $(\bar{x}_i, \bar{t}_0) = (1/2, 1/2)\,, i \in \{1,\dots,n\}$, *which does not belong to the conic MIR closure, satisfies all the nonlinear split cuts and as a result, belongs to the split closure.*



However, even in the simple case when $n = 2$, *the nonlinear split cut associated with* $\pi^3 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ *can still cut points from the side of the conic MIR closure which is illustrated in the next figure.*



Example. *Consider the second-order cone*

$$LC := \left\{(x,t_0) \in \mathbb{R}^2 \times \mathbb{R}_+ : \|B\,(x-c)\,\| \le t_0\right\}\,,$$

where $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $c = \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}$. *Also consider the disjunctions* $\pi^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\pi^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, *and* $\pi^3 = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$. *Now minimizing* t_0 *over the second-order cone after adding the cuts, the optimal value of the model with nonlinear split cuts will be larger than the optimal value of the model with conic MIR cuts with the amount of 0.0105342. Therefore, there are also instances for which nonlinear split cuts can outperform the conic MIR cuts.*

Numerical Experiments

We report some preliminary numerical results for the closest vector problem of the form

$$\text{minimize } t_0 = \|B\,(x-c)\,\|, x \in \mathbb{Z}^n.$$

Matrix B is generated uniformly at random in {-3, …, 3} and vector c is generated uniformly at random in [-1,1]. In the next table, we report the averaged gap closed (in percent), which is the amount that the integrality gap is closed after adding the cuts. In the next table, # denotes the number of instances, NSC problem denotes the original problem after adding the nonlinear split cuts, conic MIR problem denotes the original problem after adding the conic MIR cuts, El denotes the elementary cuts and NEl denotes non-elementary cuts.

n	#	NSC problem (% gap closed)		Conic MIR problem (% gap closed)	
		El	El + NEl	El	El + NEl
5	10	45.73	77.05	43.77	77.72
10	10	37.20	50.27	36.77	52.52
15	10	28.64	41.14	28.22	42.62
20	10	22.29	31.36	21.86	33.21
25	10	21.65	25.75	21.45	27.52
30	10	16	22.71	15.89	24.12
35	7	16.54	22.54	16.68	23.46
Average		26.87	38.69	26.39	41.50

Table 1: **Performance of the nonlinear split cuts and conic MIR cuts when n elementary and n non-elementary disjunctions are added**

Future Work

Nonlinear split cuts can be extended to more general conic sets such as p-order cones.

Let $p \ge 1$ be a real number and consider the p-order cone of the form

$$C_p := \{(x,t_0) \in \mathbb{R}^n \times \mathbb{R}_+ : \|x-c\|_p \le t_0\}\,,$$

where $\|x-c\|_p = \left(\sum_{i=1}^{n} |x_i - c_i|^p\right)^{1/p}$.

Proposition. *Let* $c \in \mathbb{R}^n$ *and* $k \in \{1,2,\dots,n\}$. *For* $\pi = e^k$, *where* e^k *denotes the k-th unit vector, and* $\pi_0 \in \mathbb{Z}$ *such that* $\pi_0 < c_k < \pi_0 + 1$, *we have*

$$C_p^{\pi,\pi_0} := \{(x,t_0) \in \mathbb{R}^n \times \mathbb{R}_+ : \quad \|x-c\|_p \le t_0,$$
$$\left(|\alpha\,(x_k - c_k) + \beta|^p + \sum_{i=1, i\ne k}^{n} |x_i - c_i|^p\right)^{1/p} \le t_0\},$$

where

$$\alpha = \frac{\pi_1 + \pi_0 - 2c_k}{\pi_1 - \pi_0},$$

and

$$\beta = -\frac{2\,(\pi_1 - c_k)\,(\pi_0 - c_k)}{\pi_1 - \pi_0}.$$

Moreover, nonlinear split cuts can also be extended to the quadratic sets.

Let

$$LC^2\,(B,c) := \left\{(x,t_0) \in \mathbb{R}^n \times \mathbb{R}_+ : \|B(x-c)\|^2 \le t_0\right\}.$$

Proposition. *Let* $B \in \mathbb{R}^{n\times n}$ *be an invertible matrix. Also let* $c \in \mathbb{R}^n$, $(\pi,\pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$. *We have*

$$LC^2\,(B,c)^{\pi,\pi_0} := \{(x,t_0) \in \mathbb{R}^n \times \mathbb{R}_+ : \quad \|B\,(x-c)\,\|^2 \le t_0,$$
$$\|P^{\perp}_{B^{-T}\pi}B\,(x-c)\,\|^2 + a\frac{\langle \pi, x-c\rangle}{\|B^{-T}\pi\|} + b \le t_0\},$$

where $a = \frac{\pi_1+\pi_0-2\langle\pi,c\rangle}{\|B^-T\pi\|}$ *and* $b = -\frac{(\pi_1-\langle\pi,c\rangle)(\pi_0-\langle\pi,c\rangle)}{\|B^-T\pi\|^2}$.