Mixed n-Step MIR Inequalities: Facets for the n-Mixing Set

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Research Contributions

- Introduced the n-mixing set, a new generalization of the mixing set [3] to the case where each constraint has multiple integer variables.
- Developed the mixed n-step MIR inequalities, a class of facet-defining inequalities for n-mixing set.
- New families of multi-row cuts for general MIPs
- New cuts for special structure MIPs (multi-module lot sizing, multi-module facility location)
- Successful computational results for small MIPLIB instances, random multi-module lot-sizing instances

Mixing Inequalities [3]

$$Q^{m,1} = \{(y^1,\dots,y^m,v) \in \mathbb{Z}^m \times \mathbb{R}_+ : \alpha_1 y^i + v \geqslant \beta_i, i = 1,\dots,m\}$$

- $Q^{m,1}$: mixing set, special case of the set $Q^{m,n}$
- Multi-constraint set, each constraint has 1 integer variable
- Arises in lot-sizing, facility location, network design MIPs
- Variants studied in literature: mixing set with multiple divisible/non-divisible coefficients, continuous mixing set, mixing set with flows, mixing set linked by bidirected paths

Type I and type II mixing inequalities for $K \subseteq \{1,\dots,m\}$. Let $K = \{1,\dots,k\}$ wlog where $\beta^{(1)}_{i-1} \leqslant \beta^{(1)}_i, i = 2,\dots,k$.

$$v \geqslant \sum_{i=1}^{k} \left(\beta_i^{(1)} - \beta_{i-1}^{(1)}\right)\left(\left\lceil \frac{\beta_i}{\alpha_1}\right\rceil - y^i\right)$$
$$v \geqslant \sum_{i=1}^{k} \left(\beta_i^{(1)} - \beta_{i-1}^{(1)}\right)\left(\left\lceil \frac{\beta_i}{\alpha_1}\right\rceil - y^i\right) + \left(\alpha_1 - \beta_k^{(1)}\right)\left(\left\lceil \frac{\beta_1}{\alpha_1}\right\rceil - y^1 - 1\right)$$

- Developed using the MIR inequalities associated with the constraints of $Q^{m,1}$
- Mixing inequalities describe convex hull of the mixing set
- Generate valid inequalities for single capacity lot-sizing, single capacity facility location, capacitated network design problems and general MIPs

n-Step MIR Inequalities [4]

Developed for the mixed integer knapsack set

$$Q^{1,n} = \left\{(y,v) \in \mathbb{Z} \times \mathbb{Z}_+^{n-1} \times \mathbb{R}_+ : \sum_{j=1}^{n} \alpha_j y_j + v \geqslant \beta\right\}$$

n-step MIR facet for $Q^{1,n}$:

$$\beta^{(n)} \sum_{j=1}^{n} \prod_{l=j+1}^{n} \left\lceil \frac{\beta^{(l-1)}}{\alpha_l} \right\rceil y_j + v \geqslant \beta^{(n)} \prod_{l=1}^{n} \left\lceil \frac{\beta^{(l-1)}}{\alpha_l} \right\rceil$$

Condition for validity of the n-step MIR facet (**n-step MIR conditions**):

$$\alpha_i \left\lceil \beta^{(i-1)}/\alpha_i \right\rceil \leqslant \alpha_{i-1} \quad \text{for } i = 2,\dots,n$$

Notation: For $\beta \in \mathbb{R}$, $\beta^{(j)} := \beta^{(j-1)} - \alpha_j \left\lfloor \beta^{(j-1)}/\alpha_j \right\rfloor$

Mixed n-Step MIR Inequalities [7]

- n-mixing set:

$$Q^{m,n} = \left\{(y^1,\dots,y^m,v) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+ : \sum\nolimits_{j=1}^{n} \alpha_j y_j^i + v \geqslant \beta_i, i = 1,\dots,m\right\}$$

- Wlog $K = \{1,\dots,k\}$, where $k \leqslant m$; n-step MIR conditions hold for $i = 2,\dots,k$; and ${\beta_{i-1}}^{(n)} \leqslant {\beta_i}^{(n)}, i = 2,\dots,k$.

n-step MIR inequality associated with constraint i:

$$v \geqslant {\beta_i}^{(n)} \left(\prod_{l=1}^{n} \left\lceil \frac{{\beta_i}^{(l-1)}}{\alpha_l} \right\rceil - \sum_{j=1}^{n} \prod_{l=j+1}^{n} \left\lceil \frac{{\beta_i}^{(l-1)}}{\alpha_l} \right\rceil y_j^i \right)$$

Let

$$\phi^i(y^i) = \prod_{l=1}^{n} \left\lceil \frac{{\beta_i}^{(l-1)}}{\alpha_l} \right\rceil - \sum_{j=1}^{n} \prod_{l=j+1}^{n} \left\lceil \frac{{\beta_i}^{(l-1)}}{\alpha_l} \right\rceil y_j^i$$

where $y^i = (y_1^i, \dots, y_n^i)$. Compact form of n-step MIR inequality: $v \geqslant {\beta_i}^{(n)} \phi^i(y^i)$.
Mixed n-step MIR inequalities generated by rows in K:

$$\text{Mixed } n\text{-step MIR inequalities:} \begin{cases} \text{Type I:} & v \geqslant \sum_{i=1}^{k} \left({\beta_i}^{(n)} - {\beta_{i-1}}^{(n)}\right) \phi^i(y^i) \\ \text{Type II:} & v \geqslant \sum_{i=1}^{k} \left({\beta_i}^{(n)} - {\beta_{i-1}}^{(n)}\right) \phi^i(y^i) + \left(\alpha_n - {\beta_k}^{(n)}\right)\left(\phi^1(y^1) - 1\right) \end{cases}$$

Theorem 1. The type I and type II mixed n-step MIR inequalities are valid for $Q^{m,n}$.
Consider the following variant of $Q^{m,n}$:

$$\widehat{Q}^{m,n} = \Big\{(y^1,\dots,y^m,v) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+^m : \sum_{j=1}^{n} \alpha_j y_j^i + v_i \geqslant \beta_i, i = 1,\dots,m\Big\}.$$

Corollary 1. The mixed n-step MIR inequalities with v replaced by $\overline{v}$ where $\overline{v} \geqslant v_i$ for $i \in K$, are valid for $\widehat{Q}^{m,n}$.

Facets for the n-Mixing Set

Theorem 2. The type I mixed n-step MIR inequality is facet-defining for $conv(Q^{m,n})$.

Theorem 3. The type II mixed n-step MIR inequality defines a face of dimension at least $n(m-1)$ for $conv(Q^{m,n})$.

Theorem 4. The type II mixed n-step MIR inequality is facet-defining for $conv(Q^{m,n})$ if the following additional conditions are satisfied:

(a). $\left\lfloor {\beta_1}^{(j-1)}/\alpha_j \right\rfloor \geqslant 1, j = 2,\dots,n,$

(b). ${\beta_k}^{(n)} - {\beta_1}^{(n)} \geqslant \max\left\{\alpha_{j-1} - \alpha_j \left\lceil {\beta_1}^{(j-1)}/\alpha_j \right\rceil, j = 2,\dots,n\right\}.$

Sketch of proof of Theorems 2 and 4.

- Assume that a general hyperplane passes through the face defined by the corresponding inequality.
- Find feasible points lying on this face and substitute them in the general hyperplane to derive relationships between variable coefficients.
- Reduce the general hyperplane to a scalar multiple of the support hyperplane form of the inequality.
- Proof of Theorem 3: slight modification of above approach.

Cuts for general MIP

- A general MIP set:

$$Y_m = \left\{(x_1,\dots,x_N,s) \in \mathbb{Z}_+^N \times \mathbb{R}_+^m : \sum_{t=1}^{N} a_{it} x_t + s_i \geqslant b_i, i = 1,\dots,m\right\}$$

- Given n parameters $(\alpha_1,\alpha_2,\dots,\alpha_n)$, let $K = \{1,\dots,k\}$ wlog such that ${b_{i-1}}^{(n)} \leqslant {b_i}^{(n)}, i = 2,\dots,k$ and the n-step MIR conditions hold for $i \in K$.
- We derived a function $\sigma^n_{\alpha,b} : \mathbb{R}^k \to \mathbb{R}$ based on the mixed n-step MIR facets.
- This function can be used to generate multi-row cuts for Y_m.

Theorem 5. The inequality

$$\sum_{t=1}^{N} \sigma^n_{\alpha,b}(a_t) x_t + \overline{s} \geqslant \sigma^n_{\alpha,b}(b)$$

is valid for Y_m, where $\overline{s} \geqslant s_i$ for all $i \in K$.

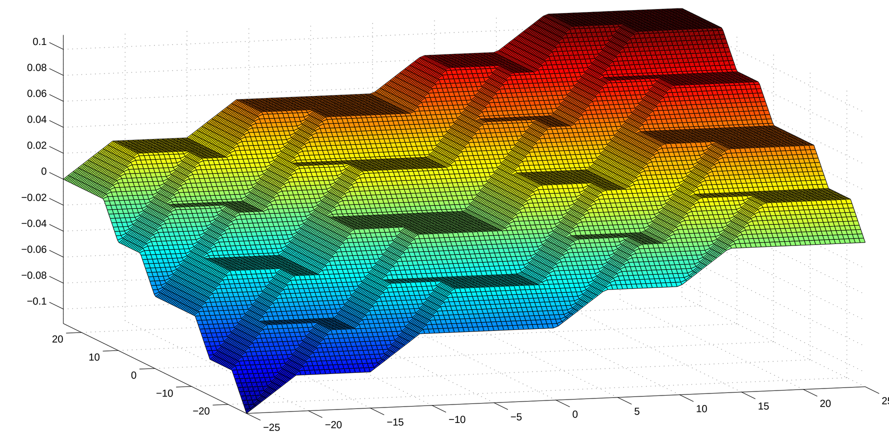

Figure: $\sigma^2_{\alpha,b}(d_1,d_2)$ over $[-25,25]^2$ with $\alpha = (25,10)$ and $b = (39,18)$

Multi-Module Lot-sizing (MML)

- Multi-module lot-sizing, a new generalization of lot-sizing:

$$\min\{\textstyle\sum_{t\in T} p_t x_t + \sum_{t\in T} h_t s_t + \sum_{t\in T} \sum_{j=1}^{n} f_t^j z_t^j : (x,s,z) \in X^{MML}\}, \text{ where}$$

$$X^{MML} = \Big\{(x,s,z) \in \mathbb{R}_+^m \times \mathbb{R}_+^m \times \mathbb{Z}_+^{m\times n} : \quad s_{t-1} + x_t = d_t + s_t, \ t \in T; \quad x_t \leqslant \textstyle\sum_{j=1}^{n} \alpha_j z_t^j, \ t \in T \Big\}$$

- $(\alpha_1,\dots,\alpha_n)$ - capacity modules
- Aggregating flow balance constraints and relaxing some x_t variables to their capacity upper bounds gives constraints like

$$s_{k-1} + \sum\nolimits_{t\in\{k,\dots,n_i-1\}\setminus S_i} x_t + \sum\nolimits_{j=1}^{n} \alpha_j \left(\sum\nolimits_{t\in S_i} z_t^j\right) \geqslant b_i$$

- Above constraints have the same form as the defining constraints of $Q^{m,n}$
- Mixed n-step MIR inequalities written based on these constraints are cuts for X^{MML}
- These mixed n-step MIR cuts generalize (k,l,S,I) inequalities of Pochet and Wolsey for lot-sizing problems with constant capacity [6]

Multi-Module Facility Location (MMF)

Multi-module facility location, a new generalization of facility location:

$$X^{MMF} = \Big\{(x,u) \in \mathbb{R}_+^{n_P n_Q} \times \{0,1\}^{n_P n} : \quad \textstyle\sum_{p\in P} x_{pq} = d_q, \ q \in Q; \quad \sum_{q\in Q} x_{pq} \leqslant \sum_{j=1}^{n} \alpha_j u_p^j, \ p \in P \Big\}.$$

- An aggregation and relaxation similar to that of X^{MML} can be performed to get constraints with a structure similar to constraints of n-mixing set.
- Mixed n-step MIR inequalities written for the above constraints are cuts for X^{MMF}.
- Mixed n-step MIR cuts for X^{MMF} generalize valid inequalities for single capacity facility location problems [1].

Computation - MIPLIB

- Three sets of experiments:
  - MIR cuts (1MIR)
  - Mixed 1-step MIR cuts over MIR cuts (1MIR1Mix)
  - Mixed 1-step MIR cuts over MIR cuts (1MIR2Mix)
- Two-row cuts, parameters - positive coefficients of integer variables
- Aggregation and bound substitution heuristics [5] used to generate base inequalities

Cut addition strategy:

- 1MIR: Add violated 1-step MIR cuts, re-optimize LP, add cuts that violate the new solution, re-optimize, and repeat until there is no improvement in LP objective.
- 1MIR1Mix: Add all possible 1-step MIR cuts according to previous procedure, and add a round of 2-row mixed 1-step MIR cuts that violate the LP solution obtained after adding MIR cuts.
- 1MIR2Mix: Same procedure as 1MIR1Mix, with 2-row mixed 2-step MIR cuts.

	Instance	flugpl	gt2	lseu	mas74	mas76	mod008	p0033	rgn
DEFAULT	*zlp*	1167190	13460.2	834.68	10482.8	38893.9	290.93	2520.57	48.8
	zmip	1201500.0	21166.0	1120.0	11801.2	40005.1	307.0	3089.0	82.2
	time	0.0	0.0	0.1	278.6	50.2	0.2	0.0	0.1
	nodes	94	1	101	2672210	403345	577	6	523
1MIR	*cuts*	0	22	47	81	116	98	28	31
	zcut	1167190.0	20592	918.9	10575.6	39024.0	298.9	2598.1	57.6
	time	0.0	0.0	0.1	305.6	19.5	0.1	0.0	0.3
	nodes	94	1	123	2764416	178778	45	1	1050
	gapclosed	0.00	92.55	29.50	7.04	11.71	49.32	13.64	26.39
1MIR1MIX	*cuts*	0	29	23	35	40	19	23	0
	zcut	1167190.0	20592.9	942.9	10580.0	39036.0	299.4	2628.2	57.6
	time	0.0	0.0	0.1	325.0	32.5	0.1	0.0	0.3
	nodes	94	1	140	2954935	281679	22	3	1050
	gapclosed	0.00	92.56	37.93	7.37	12.79	52.49	18.93	26.39
1MIR2MIX	*cuts*	0	472	75	347	138	1432	11	0
	zcut	1167190.0	20726.5	998.9	10583.1	39056.2	300.3	2636.3	57.6
	time	0.0	0.1	0.1	316.4	78.8	0.2	0.0	0.3
	nodes	94	1	132	2734233	296988	39	1	1050
	gapclosed	0.00	94.30	57.57	7.61	14.61	58.58	20.36	26.39

*gapclosed = (zcut-zlp)*100/(zmip-zlp)*

Computation - Random MML

- Four sets of randomly generated MML instances with 60 time periods.
- Two sets of capacity modules: $(180,80)$, $(270,130)$.
- Two sets of setup costs for modules: $(1000,600)$, $(5000,2600)$.
- Instance generation and separation inspired by [2].

Instance		NOCUTS				2MIX				
(α_1,α_2)	(f_t^1,f_t^2)	*zlp*	*zmip*	*time*	*nodes*	*cuts*	*zcut*	*time*	*nodes*	*gapclosed*
(180,80)	(1000,600)	559248	567703	0.3	517	729	566565	**0.4**	**73**	**86.54**
		646576	654258	0.2	506	509	653332	**0.2**	**17**	**87.95**
		615880	623663	0.1	261	443	622775	**0.1**	**1**	**88.59**
		612767	620872	0.0	58	589	620185	**0.2**	**2**	**91.52**
		571612	580115	0.2	470	607	579458	**0.1**	**1**	**92.27**
	(5000,2600)	761700	785624	109.7	508198	572	782166	**5.0**	**2534**	**85.55**
		812633	835040	53.1	228982	741	831892	**7.7**	**1942**	**85.95**
		831488	852734	61.2	240425	567	849985	**4.8**	**2603**	**87.06**
		812841	832604	30.3	145749	520	830666	**0.9**	**399**	**90.19**
		761053	782019	39.8	164846	570	780009	**1.2**	**564**	**90.41**
(270,130)	(1000,600)	730889	741886	0.0	43	488	740768	**0.2**	**22**	**89.83**
		590107	598604	0.0	29	664	597766	**0.3**	**9**	**90.14**
		616219	627391	0.3	412	578	626296	**0.2**	**1**	**90.20**
		619897	630661	0.0	18	721	629622	**0.3**	**22**	**90.35**
		541672	550644	0.0	157	458	549868	**0.1**	**1**	**91.35**
	(5000,2600)	604703	629971	19.2	86812	742	626920	**4.9**	**3288**	**87.93**
		749124	774130	2.0	6809	517	771468	**0.9**	**453**	**89.35**
		703081	726339	0.5	1161	652	724118	**0.6**	**123**	**90.45**
		660877	684319	0.6	1439	651	682235	**0.6**	**183**	**91.11**
		669220	691974	0.6	973	612	690164	**0.5**	**43**	**92.05**

Major References

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