CHANCE-CONSTRAINED BINARY PACKING PROBLEMS
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PROBLEM DESCRIPTION
A Generic Chance-Constrained Binary Packing Problem
\[ \min \left\{ cx \mid P(\bar{A}x \leq \bar{b}) \geq 1 - \epsilon, x \in \{0,1\}^n \right\} \]
- Uncertainty: nonnegative coefficient matrix \( \bar{A} \) and capacity \( \bar{b} \).
- Decisions \( x_j \): whether or not to select an item \( j \).
- Chance constraint: The packing constraints are satisfied jointly with probability at least \( 1 - \epsilon \). Interesting special cases:
  - \( \bar{A} \) is a one-row vector \( \bar{a} \): Chance-constrained knapsack problems.
  - \( \bar{A} \) is a 0-1 matrix: Chance-constrained set packing problems.

Limitations of Previous Work
- Assumption on the special structure of randomness.
- Conservative approximation.
Our Assumption
\( \bar{A} \) and \( \bar{b} \) follow a finite distribution
- For general distribution, solve a sample approximation with finite number of scenarios.

A NATURAL FORMULATION WITH SCENARIO VARIABLES

Big-M Formulation with Scenario Variables
Given a set of scenarios \( S \), each scenario happens with probability \( p_k \), \( k \in S \).
\[ \begin{align*}
\max & \quad cx \\
\text{s.t.} & \quad \sum p_k z_k \geq 1 - \epsilon \\
& \quad a^k x \leq b^k + M_k(1 - z_k), \forall k \in S \\
& \quad z \in \{0,1\}^S, \quad x \in \{0,1\}^n.
\end{align*} \]

Coefficients strengthening by Qiu et al [1]
- Update \( M_k \) by iteratively solving the LP relaxation of (2) using old \( M_k \) to model the logic constraint.
- Problem: very time-consuming.
A Scenario-based Upper Bound for \( M_k \)
- Solve for each scenario \( k' \):
\[ (\eta_k^b)^{k'} := \max \sum_{j \in N} a^k_{k'} x_j - b^k_{k'} < \epsilon \]
\[ \text{s.t.} a^k_{k'} x \leq b^k_{k'}, \quad x \in \{0,1\}^n. \]
- Another chance-constrained packing problem!
- Valid Big-M: Just an upper bound on \( M_k^* \).

NUMERICAL RESULTS

Instances Strengthened Big-M Best Cover
<table>
<thead>
<tr>
<th>( \delta )</th>
<th>#Nodes</th>
<th>#Nodes</th>
</tr>
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<td>2232</td>
</tr>
<tr>
<td>ins2</td>
<td>100</td>
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</tr>
</tbody>
</table>

Decomposed Projection

Motivation: Obtain good relaxation from formulation with scenario variables in the cover-based formulation (also true for multi-row).
- \( X_{proj} \) = projection of LP relaxation of (1) into space \( \mathcal{X} \) to be used for the cover formulation.
- \( X_{proj} \) is given by the following inequalities:
\[ \begin{align*}
\sum p_k M_k (a^k x - b^k) & \leq \epsilon, \forall S \subseteq S. \\
\end{align*} \]

A FORMULATION BASED ON PROBABILISTIC COVERS

Probabilistic covers A set \( C \) is a probabilistic cover, if \( \phi(C) > \epsilon \), where \( \phi(C) := \sum_{k \in S} p_k 1(\sum_{j \in C} a_j \notin \bar{b}) \) is the probability of violation.
A formulation based on probabilistic covers
\[ \sum_{j \in C} x_j \leq |C| - 1, \forall \text{probabilistic cover } C \]
- No extra scenario variables.
- Finite, but exponentially many constraints.
Leverage techniques for cover-based algorithms in deterministic MIP
- Minimize cover inequalities are only facet-defining when \( x_j = 0, \forall j \notin C \).
- Lifting: Derive valid coefficients for \( x_j, j \notin C \) by solving a series of optimization problems.

A Natural Formulation with Scenario Variables

Proabilistic lifting
- Consider the first variable \( x_i \) to be uplifted (lift from 0) in the sequence:
\[ \begin{align*}
\bar{G} &= \max \sum_{j \in C} x_j \\
\text{s.t.} & \quad \sum_{k \in S} p_k 1(\sum_{j \in C} a_j^k x_j \leq b^k - a_i^k) \geq 1 - \epsilon \\
& \quad x \in \{0,1\}^{|C|}.
\end{align*} \]
- Lifting coefficient \( \bar{f}(i) = \max \{|C| - 1 - \bar{G}, 0\} \).
- Similar for downlifting: lift from 1.
- Only need an upper bound of (6) for a valid lifting coefficient. \( \Rightarrow \) Again, use the scenario-based upper bound for a lifting coefficient.

Enhancement 1: warm start
- Implementation: Zemel’s Algorithm ([2]).
- Idea: warmstart lifting using information from the previous lifting problem.
- Calculate scenario-based upper bound \( \zeta_k \) independently for each scenario \( k \), but update the basis using the same approximated lifting coefficient \( \beta_k \).

Enhancement 2: local cuts
- A restricted set with some variables fixed.
- Get the most valid inequality for the restricted set by:
  - Enumerating all non-dominated feasible solutions.
  - Solving the polar CGLP.
- Motivation: better initial valid inequality.

REFERENCE

Get the most violated valid inequality for the restricted set by:
- Enumerating all non-dominated feasible solutions.
- Solving the polar CGLP.
- Motivation: better initial valid inequality.