Chance-constrained Binary Packing Problems

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Problem Description

A Generic Chance-Constrained Binary Packing Problem

$$\min\{cx \mid \mathbb{P}\{\tilde{A}x \le \tilde{b}\} \ge 1-\epsilon,\ x \in \{0,1\}^n\}$$

- Uncertainty: nonnegative coefficient matrix $\tilde{A}$ and capacity $\tilde{b}$.
- Decisions x_j: whether or not to select an item j.
- Chance constraint: The packing constraints are satisfied *jointly* with probability at least $1-\epsilon$.
- Interesting special cases:
  - $\tilde{A}$ is a one-row vector $\tilde{a}$: Chance-constrained knapsack problems.
  - $\tilde{A}$ is a 0-1 matrix: Chance-constrained set packing problems.

Limitations of Previous Work

- Assumption on the special structure of randomness.
- Conservative approximation.

Our Assumption

$\tilde{A}$ and $\tilde{b}$ follow a finite distribution

- For general distribution, solve a sample approximation with finite number of scenarios.

A Natural Formulation with Scenario Variables

Big-M Formulation with Scenario Variables

Given a set of scenarios S, each scenario happens with probability $p_k, k \in S$.

$$\begin{aligned} \max\ & cx \\ \text{s.t.}\ & \sum_{k\in S} p_k z_k \ge 1-\epsilon \qquad (1) \\ & a^k x \le b^k + M^k(1-z_k), \forall k \in S \\ & z \in \{0,1\}^{|S|}, x \in \{0,1\}^n. \end{aligned}$$

- Naive Big-M coefficients may lead to weak relaxation bounds.

Big-M coefficients strengthening

$$M^k \ge M_*^k := \max_{x\in\{0,1\}^n} \sum_{j\in N} a_j^k x_j - b_i^k \qquad (2)$$
$$\text{s.t.} \sum_{k'\in S} p_{k'} \mathbf{1}(a^{k'}x \le b^{k'}) \ge 1-\epsilon$$

- Another chance-constrained packing problem!
- Valid Big-M: Just an upper bound on M_*^k.

Coefficient strengthening by Qiu et al [1]

- Update M^k by iteratively solving the LP relaxation of (2) using old M^k to model the logic constraint.
- Problem: very time-consuming.

A Scenario-based Upper Bound for M^k

- Solve for each scenario k':

$$(\eta^k)_{k'} := \max \sum_{j\in N} a_j^k x_j - b_i^k \qquad (3)$$
$$\text{s.t. } a^{k'}x \le b^{k'},\ x \in \{0,1\}^n$$

- Sort $(\eta^k)_1 \ge (\eta^k)_2 \ge \cdots \ge (\eta^k)_{|S|}$.
- Let $q = \max\{k \mid \sum_{j=1}^k p_j \le \epsilon\}$, then $(\eta_i^k)_{q+1}$ gives an upper bound on M_*^k.
- A relaxation of (3) is sufficient for a valid upper bound $\Rightarrow$ efficient!

Instances		IterLP		Scen	
	$\lvert S\rvert$	Pre-T	Sol-T	Pre-T	Sol-T
ins1	100	1.5	0.5	0.1	0.3
	1000	696.8	27.1	4.8	14.6
ins2	100	2.3	0.9	0.1	0.8
	1000	1093.1	713.2	8.7	567.5

Decomposed Projection

Motivation: Obtain good relaxation from formulation with scenario variables in the cover-based formulation (also true for multi-row).

- X_{proj} = projection of LP relaxation of (1) into x space $\Rightarrow$ can be used for the cover formulation.
- X_{proj} is given by the following inequalities:

$$a^k x \le b^k + M^k, \forall k \in S \qquad (4)$$
$$\sum_{k\in\bar{S}} \frac{p_k}{M^k}(a^k x - b^k) \le \epsilon, \forall \bar{S} \subseteq S. \qquad (5)$$

- Separation of (5) is easy.

Numerical Results

Instances		No lifting		Lifting	
	$\lvert S\rvert$	Time	Gap	Time	Gap
ins1	100	>2505(2)	1.9%	64	-
(n=50)	1000	>3319(2)	0.8%	134	-
	3000	>3600(0)	0.6%	382	-
ins2	100	>3600(0)	2.3%	>3508(1)	0.6%
(n=90)	1000	>3600(0)	2.1%	>3600(0)	0.7%
	3000	>3600(0)	2.0%	>3600(0)	0.9%

- Lifting is indispensible.

Instances		+local		+local & proj	
	$\lvert S\rvert$	Time	# Nodes	Time	# Nodes
ins1	100	51	1253	29	1121
(n=50)	1000	211	1228	172	1243
	3000	606	1783	800	2117
ins2	100	509	2292	36	399
(n=90)	1000	>2497(3)	>3180	287	419
	3000	>3400(1)	>1740	849	446

- Local cuts and decomposed projection are useful.

Instances		Strengthened Big-M		Best Cover	
	$\lvert S\rvert$	Time	# Nodes	Time	# Nodes
ins1	100	0.5	370	29	1121
(n=50)	1000	27	6570	172	1243
	3000	>2829(2)	>129792	800	2117
ins2	100	1	690	36	399
(n=90)	1000	712	85745	287	419
	3000	>3600(0)	>68440	849	446

- Formulation (1) does not scale well with # of scenarios.

A Formulation Based on Probabilistic Covers

Probabilistic covers A set C is a probabilistic cover, if $\phi(C) > \epsilon$, where $\phi(C) := \sum_{k\in S} p_k \mathbf{1}\{\sum_{j\in C} \tilde{a}_j \not\le \tilde{b}\}$ is the probability of violation.

A formulation based on probabilistic covers

$$\sum_{j\in C} x_j \le |C| - 1, \forall \text{ probabilistic cover } C$$

- No extra scenario variables.
- Finite, but exponentially many constraints.

Leverage techniques for cover-based algorithms in deterministic MIP

- Minimal cover inequalities are only facet-defining when $x_j = 0, \forall j \notin C$.
- Lifting: Derive valid coefficients for $x_j, j \notin C$ by solving a series of optimization problems.

Probabilistic Lifting

- Consider the first variable x_t to be uplifted (lift from 0) in the sequence:

$$\begin{aligned} \zeta_t^* = \max\ & \sum_{j\in C} x_j \qquad (6) \\ \text{s.t.}\ & \sum_{k\in S} p_k \mathbf{1}\Big(\sum_{j\in C} a_j^k x_j \le b^k - a_t^k\Big) \ge 1-\epsilon \\ & x \in \{0,1\}^{|C|}. \end{aligned}$$

- Lifting coefficient $\beta_t^* = \max\{|C| - 1 - \zeta_t^*, 0\}$.
- Similar for downlifting: lift from 1.
- Only need an upper bound of (6) for a valid lifting coefficient. $\Rightarrow$ Again, use the scenario-based upper bound for a lifting coefficient.

Enhancement 1: warm start

- Implementation: Zemel's Algorithm ([2]). Idea: warmstart lifting using information from the previous lifting problem.
- Calculate scenario-based upper bound ζ_t^k independently for each scenario k, but update the basis using the same approximated lifting coefficient β_t.

Enhancement 2: local cuts

- A restricted set with some variables fixed.
- Get the most violated valid inequality for the restricted set by:
  - Enumerating all non-dominated feasible solutions.
  - Solving the polar CGLP.
- Motivation: better initial valid inequality.

Reference

[1] F. Qiu, S. Ahmed, S. Dey, and L. Wolsey. Covering Linear Programming with Violations. Submitted, 2012.

[2] E. Zemel. Easily Computable Facets of the Knapsack Polytope. Mathematics of Operations Research, volume 14, 1989, page 760–764.