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Crystals from categorified quantum groups

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Abstract

We study the crystal structure on categories of graded modules over algebras which categorify the negative half of the quantum Kac–Moody algebra associated to a symmetrizable Cartan data. We identify this crystal with Kashiwara's crystal for the corresponding negative half of the quantum Kac–Moody algebra. As a consequence, we show the simple graded modules for certain cyclotomic quotients carry the structure of highest weight crystals, and hence compute the rank of the corresponding Grothendieck group.

Keywords: Crystals; Categorification; Khovanov–Lauda–Rouquier algebras; Quiver Hecke algebras; Quantum groups

Contents

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1. Introduction

In [31,33,53] a family R of graded algebras was introduced that categorifies the integral form ${}_{\mathcal{A}}\mathbf{U}_q^- := {}_{\mathcal{A}}\mathbf{U}_q^-(\mathfrak{g})$ of the negative half of the quantum enveloping algebra $\mathbf{U}_q(\mathfrak{g})$ associated to a symmetrizable Kac–Moody algebra $\mathfrak{g}$. The grading on these algebras equips the Grothendieck group $K_0(R\text{-pmod})$ of the category of finitely-generated graded projective R-modules with the structure of a $\mathbb{Z}[q, q^{-1}]$-module, where $q^r[M] := [M\{r\}]$, and $M\{r\}$ denotes a graded module M with its grading shifted up by r. Natural parabolic induction and restriction functors give $K_0(R\text{-pmod})$ the structure of a (twisted) $\mathbb{Z}[q, q^{-1}]$-bialgebra. In [31,33] an explicit isomorphism of twisted bialgebras was given between ${}_{\mathcal{A}}\mathbf{U}_q^-$ and $K_0(R\text{-pmod})$. The crystal-theoretic methods in this paper provide a new proof of this result.

Several conjectures were also made in [31,33]. One conjecture that was unproven at the time this article first appeared is the so-called cyclotomic quotient conjecture which suggests a close connection between certain finite dimensional quotients of the algebras R and the integrable representation theory of quantum Kac–Moody algebras. At that time, the conjecture had been proven in finite and affine type A by Brundan and Kleshchev [10], but very little was known in the case of an arbitrary symmetrizable Cartan datum. By obtaining new results on the fine structure of simple R-modules, here we show that simple graded modules for these cyclotomic quotients carry the structure of highest weight crystals. Hence we identify the rank of the corresponding Grothendieck group with the rank of the integral highest weight representation, proving a major component of the cyclotomic quotient conjecture. Before this article went to press, proofs of the full conjecture appeared independently in work of Webster [60] and Kang and Kashiwara [23].

To explain these results more precisely, suppose we are given a symmetrizable Cartan datum where I is the index set of simple roots. The algebras R have a diagrammatic description and are determined by the symmetrizable Cartan datum of $\mathfrak{g}$ together with some extra parameters. In the literature these algebras are sometimes called Khovanov–Lauda–Rouquier algebras and quiver Hecke algebras.

For each $\nu \in \mathbb{N}[I]$ the block $R(\nu)$ of the algebra R admits a finite dimensional quotient $R^{\Lambda}(\nu)$ associated to the highest weight Λ, called a cyclotomic quotient. These quotients were conjectured in [31,33] to categorify the ν-weight space of the integral version of the irreducible representation $V(\Lambda)$ of highest weight Λ for $\mathbf{U}_q(\mathfrak{g})$, in the sense that there should be an isomorphism

$$V(\Lambda)_{\mathbb{C}} \overset{\cong}{\to} \bigoplus_{\nu\in\mathbb{N}[I]} K_0\big(R^{\Lambda}(\nu)\text{-pmod}\big)_{\mathbb{C}},$$

where $K_0(R^{\Lambda}(\nu)\text{-pmod})_{\mathbb{C}}$ denotes the complexified Grothendieck group of the category of graded finitely generated projective $R^{\Lambda}(\nu)$-modules. A special case of this conjecture was proven in type A by Brundan and Stroppel [12]. The more general conjecture was proven in finite and affine type A by Brundan and Kleshchev [9,10]. They constructed an isomorphism

$$R^{\Lambda}(\nu) \overset{\cong}{\to} H^{\Lambda}_{\nu},$$

where H^{Λ}_{ν} is a block of the cyclotomic affine Hecke algebra H^{Λ}_m as defined in [5,8,13]. This isomorphism induces a new grading on blocks of the cyclotomic affine Hecke algebra. This has led to the definition of graded Specht modules for cyclotomic Hecke algebras [11], the construction of a homogeneous cellular basis for the cyclotomic quotients $R^{\Lambda}(\nu)$ in type A [22], the introduction of gradings in the study of q-Schur algebras [4], and an extension of the generalized LLT conjecture to the graded setting [10].

Ariki's categorification theorem gave a geometric proof that the sum of complexified Grothendieck groups of cyclotomic Hecke algebras H^{Λ}_m at an N-th root of unity over $\mathbb{C}$, taken over all $m \geqslant 0$, was isomorphic to the highest weight representation $V(\Lambda)$ of $U(\widehat{\mathfrak{sl}}_N)$ [1], see [2,3,6,48] and also [16,40]. Grojnowski gave a purely algebraic proof of this result, parameterizing the simple H^{Λ}_m-modules in terms of crystal data of highest weight crystals [17].

Brundan and Kleshchev's proof of the cyclotomic quotient conjecture in type A utilized the isomorphism between the graded algebras $R^{\Lambda}(\nu)$ and blocks of the cyclotomic affine Hecke algebra, allowing them to extend Grojnowski's crystal theoretic classification of simples of the ungraded affine Hecke algebra to the graded setting. By keeping careful track of the gradings, they were able to extend Ariki's theorem to the graded setting, thereby proving the cyclotomic quotient conjecture in type A, as well as identifying the indecomposable projective modules for R^{Λ}_{ν} with the canonical basis for $V(\Lambda)$. Indeed, the algebras $R^{\Lambda}(\nu)$ were originally called cyclotomic quotients in [31] because they were expected to categorify irreducible highest weight representations of quantum Kac–Moody algebras analogous to the way that cyclotomic Hecke algebras categorify irreducible highest weight representations for type A in the non-quantum setting. In this way, these diagrammatically defined cyclotomic quotients can be viewed as graded extensions of the cyclotomic Hecke algebras to all types.

While there are natural extensions of cyclotomic Hecke algebras of type A, namely quotients of affine Hecke algebras of crystallographic type, they do not provide analogous categorification results. However, categorification results of a different flavour do exist in types B and D, see [56,55,15,29].

In type A homogeneous cellular bases were constructed [39,12,22]. However, the study of cyclotomic quotients outside of type A has been hindered by the lack of explicit bases for the algebras $R^{\Lambda}(\nu)$. Some explicit calculations of cyclotomic quotients $R^{\Lambda}(\nu)$ in other type were made for level one and two representations [54], but it is not clear how to extend these results to all representations. The algebras $R(\nu)$ have a PBW basis that aid in computations. No such basis is known for the algebras $R^{\Lambda}(\nu)$.

In the symmetric case the algebras R are related to Lusztig's geometric categorification using perverse sheaves. Following Ringel [52], Lusztig gave a geometric interpretation of $\mathbf{U}_q^- = \mathbf{U}_q^-(\mathfrak{g})$ [43–45], see also [46,47]. This gave rise to Lusztig's canonical basis for $\mathbf{U}_q^-$. Kashiwara defined a crystal basis of $\mathbf{U}_q^-$ for certain simple Lie algebras [25] and later proved its existence for all symmetrizable Kac–Moody algebras [26,24]; the affine type A case was proven by Misra and Miwa [49]. Kashiwara also constructed the so-called global crystal basis of $\mathbf{U}_q^-$ [26,27,24]. Grojnowski and Lusztig [18] proved that the global crystal basis and the canonical basis are the same. The canonical basis of $\mathbf{U}_q^-$ is a basis with remarkable positivity and integrality properties, and gives rise to bases in all irreducible integrable $\mathbf{U}_q(\mathfrak{g})$-representations.

Varagnolo and Vasserot constructed an isomorphism between Ext-algebras of certain simple perverse sheaves on Lusztig quiver varieties [57] and the algebras $R(\nu)$ in the symmetric case, proving a conjecture from [31]. Consequently, one can identify indecomposable projectives for the algebras R with simple perverse sheaves on Lusztig quiver varieties and the canonical basis for ${}_{\mathcal{A}}\mathbf{U}_q^-$. Rouquier has also announced a similar result.

One should be able to deduce a classification of graded simple modules for the algebras $R^{\Lambda}(\nu)$ in the symmetric case using results of [31] and [57] together with Kashiwara and Saito's geometric construction of crystals [30], but the details of this argument have not appeared. We expect that cyclotomic quotients $R^{\Lambda}(\nu)$ should also have a geometric interpretation in terms of Nakajima quiver varieties [50].

In this paper we determine the size of the Grothendieck group for arbitrary cyclotomic quotients $R^{\Lambda}(\nu)$ associated to a symmetrizable Cartan datum. Rather than working geometrically, our methods are based strongly on the algebraic treatment of the affine Hecke algebra and its cyclotomic quotients introduced by Grojnowski [17]. This approach extended Kleshchev's results for the symmetric groups [34–36], and utilizes earlier results of Vazirani [58,59] and Grojnowski and Vazirani [19]. Kleshchev's book contains an excellent exposition of Grojnowski's approach in the context of degenerate affine Hecke algebras [37]. The idea is to introduce a crystal structure on categories of modules, interpreting Kashiwara operators module theoretically. To apply this approach to the study of algebras $R(\nu)$, rather than working with projective modules, one must work with the category of finite dimensional graded $R(\nu)$-modules. This could be done by working over an algebraically closed field $\Bbbk$ and utilizing the $\mathbb{Z}[q,q^{-1}]$-bilinear pairing

$$(\,,\,): K_0\big(R(\nu)\text{-pmod}\big) \times G_0\big(R(\nu)\text{-fmod}\big) \to \mathbb{Z}\big[q,q^{-1}\big], \tag{1.1}$$

where $G_0(R(\nu)\text{-fmod})$ denotes the Grothendieck group of the category of finite dimensional graded $R(\nu)$-modules. Since the pairing is a perfect pairing (see [31]), it allows one to deduce that Serre relations hold on $G_0(R)$ from the corresponding result for $K_0(R)$. Here, however, we take a more direct approach giving a direct proof of Serre relations on $G_0(R)$ and a more direct identification of $G_0(R)$ with ${}_{\mathcal{A}}\mathbf{U}_q^-$. This is a byproduct of our careful analysis, which additionally yields new results on the structure of simple modules.

We study the crystal graph whose nodes are the graded simple $R(\nu)$-modules (up to grading shift) taken over all $\nu \in \mathbb{N}[I]$. By identifying this crystal graph with the Kashiwara crystal $B(\infty)$ associated to $\mathbf{U}_q^-$ we are able to define a crystal structure on the set of graded simple modules for the cyclotomic quotients $R^{\Lambda}(\nu)$ and show that it is the crystal graph $B(\Lambda)$. This allows us to view cyclotomic quotients of the algebras $R(\nu)$ as a categorification of the integrable highest weight representation $V(\Lambda)$ of $\mathbf{U}_q^+$, proving *part* of the cyclotomic quotient conjecture from [31] in the general setting. This does not prove the entire cyclotomic quotient conjecture as our isomorphism is only an isomorphism of $\mathbf{U}_q^+$-modules, not of $\mathbf{U}_q(\mathfrak{g})$-modules.

The study of KLR algebras and their cyclotomic quotients is rapidly developing. On the same day that this posted to the arXiv, an article by Kleshchev and Ram [38] also appeared where they construct all irreducible representations of algebras $R(\nu)$ in finite type from Lyndon words. Their work generalizes the fundamental work of [7,61] who parameterized and constructed the simple modules for the affine Hecke algebra in type A with generic parameter in terms of $\mathbf{U}^-(\mathfrak{gl}_\infty)$. Furthermore, some time after this article appeared alternative proofs of the full cyclotomic quotient conjecture were given by Webster [60] and by Kang and Kashiwara [23]. Kang and Kashiwara show that functors lifting the action of E_i and F_i in $\mathbf{U}_q(\mathfrak{g})$ are biadjoint, showing that cyclotomic quotients categorify $V(\Lambda)$ as $\mathbf{U}_q(\mathfrak{g})$-modules and give a 2-representation in the sense of Rouquier [53]. Webster's work gives a different proof of biadjointness and also constructs an action of the 2-category $\dot{\mathcal{U}}$ from [41,31] on categories of modules over cyclotomic quotients.

This article gives a proof of the crystal version of the cyclotomic quotient conjecture. This work differs from the articles mentioned above in that it requires a detailed study of the fine structure simple modules for cyclotomic quotients. We feel that this fine structure constitutes the main results obtained in this article. These results are strong enough to give an alternative proof of the categorification theorem of [31,33] staying entirely in the category of finitely-generated modules, see Section 6.3.1.

All of the results in this paper should extend to Rouquier's version of algebras $R(\nu)$ associated to Hermitian matrices, at least for those Hermitian matrices leading to graded algebras. We also believe that these results will fit naturally within Khovanov and Lauda's framework of categorified quantum groups [41,32], as well as Rouquier's 2-representations of 2-Kac–Moody algebras [14,53].

We end the introduction with a brief outline of the article, highlighting other results to be found herein. In Section 1.1 we review the definition and key properties of the algebras $R(\nu)$. In Section 2 we study various functors defined on the categories of graded modules over the algebras $R(\nu)$. In particular, Section 2.3 introduces the co-induction functor and proves several key results. In Section 3 we look at the morphisms induced by these functors on the Grothendieck rings.

Section 4 contains a brief review of crystal theory. Of key importance is the result of Kashiwara and Saito [30], recalled in Section 4.2, characterizing the crystal $B(\infty)$. In Section 5 we introduce crystal structures on the category of modules over algebras $R(\nu)$ and their cyclotomic quotients $R^{\Lambda}(\nu)$. After a detailed study of this crystal data in Section 6, these crystals are identified as the crystals $B(\infty)$ and $B(\Lambda)$ in Section 7.

1.1. The algebras $R(\nu)$

1.1.1. Cartan datum

Assume we are given a Cartan data

P – a free $\mathbb{Z}$-module (called the weight lattice),
I – an index set for simple roots,
$\alpha_i \in P$ for $i \in I$ called simple roots,
$h_i \in P^\vee = \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$ called simple coroots,
$(\,,\,) : P \times P \to \mathbb{Z}$ a bilinear form,

where we write $\langle \cdot,\cdot \rangle : P^\vee \times P \to \mathbb{Z}$ for the canonical pairing. This data is required to satisfy the following axioms

$$(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0} \quad \text{for any } i \in I, \tag{1.2}$$

$$\langle h_i, \lambda \rangle = 2\frac{(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \quad \text{for } i \in I \text{ and } \lambda \in P, \tag{1.3}$$

$$(\alpha_i, \alpha_j) \leqslant 0 \quad \text{for } i, j \in I \text{ with } i \neq j. \tag{1.4}$$

Hence $\{\langle h_i, \alpha_j \rangle\}_{i,j\in I}$ is a symmetrizable generalized Cartan matrix. In what follows we write

$$a_{ij} = -\langle i, j \rangle := -\langle h_i, \alpha_j \rangle \tag{1.5}$$

for $i, j \in I$.

Let $\Lambda_i \in P^+$ be the fundamental weights defined by $\langle h_j, \Lambda_i \rangle = \delta_{ij}$.

1.1.2. The algebra $\mathbf{U}_q^-$

Associated to a Cartan datum one can define an algebra $\mathbf{U}_q^-$, the quantum deformation of the universal enveloping algebra of the "lower-triangular" subalgebra of a symmetrizable Kac–Moody algebra $\mathfrak{g}$. Our discussion here follows Lusztig [46].

Let $q_i = q^{\frac{(\alpha_i,\alpha_i)}{2}}$, $[a]_i = q_i^{a-1} + q_i^{a-3} + \cdots + q_i^{1-a}$, $[a]_i! = [a]_i[a-1]_i \ldots [1]_i$. Denote by $'\mathbf{f}$ the free associative algebra over $\mathbb{Q}(q)$ with generators θ_i, $i \in I$, and introduce q-divided powers $\theta_i^{(a)} = \theta_i^a/[a]_i!$. The algebra $'\mathbf{f}$ is $\mathbb{N}[I]$-graded, with θ_i in degree i. The tensor square $'\mathbf{f} \otimes {'\mathbf{f}}$ is an associative algebra with twisted multiplication

$$(x_1 \otimes x_2)\big(x_1' \otimes x_2'\big) = q^{-|x_2|\cdot|x_1'|} x_1 x_1' \otimes x_2 x_2'$$

for homogeneous x_1, x_2, x_1', x_2'. The assignment $r(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i$ extends to a unique algebra homomorphism $r : {'\mathbf{f}} \to {'\mathbf{f}} \otimes {'\mathbf{f}}$.

The algebra $'\mathbf{f}$ carries a $\mathbb{Q}(q)$-bilinear form determined by the conditions

- $(1, 1) = 1$,
- $(\theta_i, \theta_j) = \delta_{i,j}(1 - q_i^2)^{-1}$ for $i, j \in I$,
- $(x, yy') = (r(x), y \otimes y')$ for $x, y, y' \in {'\mathbf{f}}$,
- $(xx', y) = (x \otimes x', r(y))$ for $x, x', y \in {'\mathbf{f}}$.

The bilinear form $(\,,\,)$ is symmetric. Its radical $\mathfrak{I}$ is a two-sided ideal of $'\mathbf{f}$. The form $(\,,\,)$ descends to a non-degenerate form on the associative $\mathbb{Q}(q)$-algebra $\mathbf{f} = {}'\mathbf{f}/\mathfrak{I}$.

Theorem 1.1. *The ideal $\mathfrak{I}$ is generated by the elements*

$$\sum_{r+s=a_{ij}+1} (-1)^r \theta_i^{(r)} \theta_j \theta_i^{(s)}$$

over all $i, j \in I$, $i \neq j$.

For a general Cartan datum, the only known proof of this theorem requires Lusztig's geometric realization of $\mathbf{f}$ via perverse sheaves. This proof is given in his book [46, Theorem 33.1.3]. Less sophisticated proofs exist when the Cartan datum is finite.

Remark 1.2. Theorem 1.1 implies that $\mathbf{f}$ is the quotient of $'\mathbf{f}$ by the quantum Serre relations

$$\sum_{r+s=a_{ij}+1} (-1)^r \theta_i^{(r)} \theta_j \theta_i^{(s)} = 0. \tag{1.6}$$

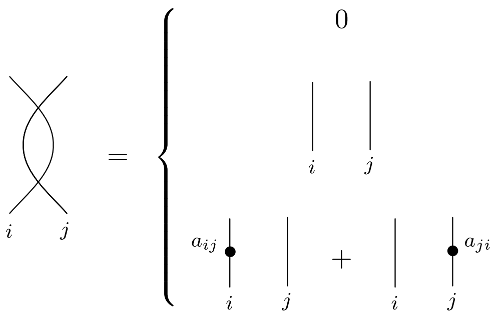
Furthermore, since $\mathbf{f}$ is an $\mathbb{N}[I]$-graded quotient of a free algebra, it also implies that there are no smaller degree relations in $\mathbf{f}$. In particular, (1.6) can never hold for $r + s = c + 1$ with $c < a_{ij}$.

Let $\mathbf{U}_q(\mathfrak{g})$ denote the quantum enveloping algebra of a symmetrizable Kac–Moody algebra $\mathfrak{g}$. There is a pair of injective algebra homomorphisms $\mathbf{f} \to \mathbf{U}_q(\mathfrak{g})$, which sends $\theta_i \mapsto e_i$, respectively $\theta_i \mapsto f_i$. We denote the images of these homomorphisms as $\mathbf{U}_q^+(\mathfrak{g})$ and $\mathbf{U}_q^-(\mathfrak{g})$. Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. The integral form of the algebra $\mathbf{f}$, denoted ${}_{\mathcal{A}}\mathbf{f}$, is the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $\mathbf{f}$ generated by the divided powers $\theta_i^{(a)}$, over all $i \in I$ and $a \in \mathbb{N}$. We write ${}_{\mathcal{A}}\mathbf{U}_q^-$ for the corresponding integral form of the negative half of the quantum enveloping algebra $\mathbf{U}_q(\mathfrak{g})$. The algebra ${}_{\mathcal{A}}\mathbf{f}$ admits a decomposition into weight spaces ${}_{\mathcal{A}}\mathbf{f} = \bigoplus_{\nu \in \mathbb{N}[I]} {}_{\mathcal{A}}\mathbf{f}(\nu)$.

In the next section we introduce graded algebras $R(\nu)$ whose Grothendieck ring was shown by Khovanov and Lauda to be isomorphic to ${}_{\mathcal{A}}\mathbf{f}$ as bialgebras, see Theorem 3.1.

1.1.3. The definition of the algebra $R(\nu)$

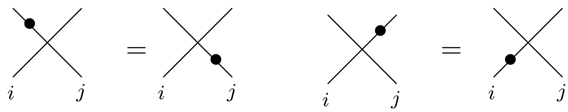
Recall the definition from [31,33] of the algebra R associated to a Cartan datum. Let $\Bbbk$ be an algebraically closed field (of arbitrary characteristic). The algebra R is defined by finite $\Bbbk$-linear combinations of braid-like diagrams in the plane, where each strand is coloured by a vertex $i \in I$. Strands can intersect and can carry dots; however, triple intersections are not allowed. Diagrams are considered up to planar isotopy that do not change the combinatorial type of the diagram. We recall the local relations



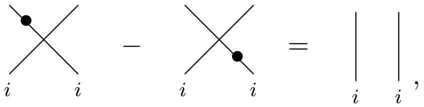
if $i = j$,

if $(\alpha_i, \alpha_j) = 0$, (1.7)

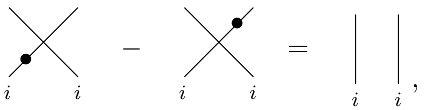
if $(\alpha_i, \alpha_j) \neq 0$,



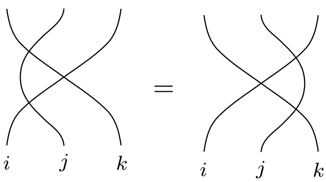
for $i \neq j$, (1.8)



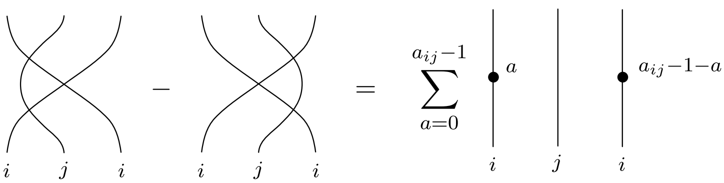
(1.9)



(1.10)



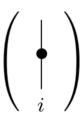
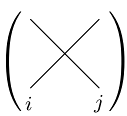
unless $i = k$ and $(\alpha_i, \alpha_j) \neq 0$, (1.11)



if $(\alpha_i, \alpha_j) \neq 0$.

(1.12)

Left multiplication is given by concatenating a diagram on top of another diagrams when the corresponding endpoints have the same colours, and is defined to be zero otherwise. The algebra is graded where generators are defined to have degrees

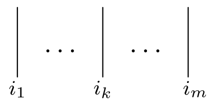
deg  $= (\alpha_i, \alpha_i)$, deg  $= -(\alpha_i, \alpha_j)$. (1.13)

For $\nu = \sum_{i \in I} \nu_i \cdot i \in \mathbb{N}[I]$ let $\mathrm{Seq}(\nu)$ be the set of all sequences of vertices $\boldsymbol{i} = i_1 \dots i_m$ where $i_r \in I$ for each r and vertex i appears ν_i times in the sequence. The length m of the sequence is equal to $|\nu| = \sum_{i\in I} \nu_i$. It is sometimes convenient to identify $\nu = \sum_{i\in I} \nu_i \cdot i \in \mathbb{N}[I]$ as $\nu \in \sum_{i\in I} \nu_i\alpha_i \in Q_+ = \bigoplus_{i\in I} \mathbb{Z}_{\geqslant 0}\alpha_i$. We denote $Q_- = -Q_+ = \bigoplus_{i\in I} \mathbb{Z}_{\leqslant 0}\alpha_i$. The algebra R has a decomposition

$$R = \bigoplus_{\nu\in\mathbb{N}[I]} R(\nu) \tag{1.14}$$

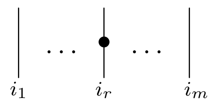
where $R(\nu)$ is the subalgebra generated by diagrams that contain ν_i strands coloured i.

To convert from graphical to algebraic notation write

$1_{\boldsymbol{i}} \quad :=$ 

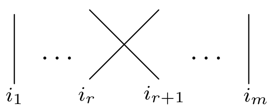
(1.15)

for $\boldsymbol{i} = i_1 i_2 \dots i_m \in \mathrm{Seq}(\nu)$. The elements $1_{\boldsymbol{i}}$ are idempotents in the ring $R(\nu)$ and when I is finite, $1_\nu \in R(\nu)$ is given by $1_\nu = \sum_{\boldsymbol{i}\in\mathrm{Seq}(\nu)} 1_{\boldsymbol{i}}$. For $1 \leqslant r \leqslant m$ we denote

$x_{r,\boldsymbol{i}} \quad :=$ 

(1.16)

with the dot positioned on the r-th strand counting from the left, and

$\psi_{r,\boldsymbol{i}} \quad :=$ 

. (1.17)

The algebra $R(\nu)$ decomposes as a vector space

$$R(\nu) = \bigoplus_{\boldsymbol{i},\boldsymbol{j}\in\mathrm{Seq}(\nu)} 1_{\boldsymbol{j}} R(\nu) 1_{\boldsymbol{i}} \tag{1.18}$$

where $1_{\boldsymbol{j}} R(\nu) 1_{\boldsymbol{i}}$ is the $\Bbbk$-vector space of all linear combinations of diagrams with sequence $\boldsymbol{i}$ at the bottom and sequence $\boldsymbol{j}$ at the top modulo the above relations.

The symmetric group S_m acts on $\mathrm{Seq}(\nu)$, $m = |\nu|$ by permutations. Transposition $s_r = (r, r+1)$ switches entries i_r, i_{r+1} of $\boldsymbol{i}$. Thus, $\psi_{r,\boldsymbol{i}} \in 1_{s_r(\boldsymbol{i})} R(\nu) 1_{\boldsymbol{i}}$. For each $w \in S_m$ fix once and for all a reduced expression $\widehat{w} = s_{w_1} s_{w_2} \dots s_{w_t}$. Given $w \in S_n$ we convert its reduced expression $\widehat{w}$ into an element of $1_{w(\boldsymbol{i})} R(\nu) 1_{\boldsymbol{i}}$ denoted $\psi_{\widehat{w},\boldsymbol{i}} = \psi_{w_1, s_{w_2}\dots s_{w_t}(\boldsymbol{i})} \cdots \psi_{w_{t-1}, s_{w_t}(\boldsymbol{i})} \psi_{w_t,\boldsymbol{i}}$. To simplify notation we introduce elements

$$x_r := \sum_{\boldsymbol{i}\in\mathrm{Seq}(\nu)} x_{r,\boldsymbol{i}}, \qquad \psi_{\widehat{w}} = \sum_{\boldsymbol{i}\in\mathrm{Seq}(\nu)} \psi_{\widehat{w},\boldsymbol{i}} \tag{1.19}$$

so that $x_r 1_{\boldsymbol{i}} = 1_{\boldsymbol{i}} x_r = x_{r,\boldsymbol{i}}$ and $\psi_{\widehat{w}} 1_{\boldsymbol{i}} = 1_{w(\boldsymbol{i})} \psi_{\widehat{w}} = \psi_{\widehat{w},\boldsymbol{i}}$. This allows us to write the definition of the algebra $R(\nu)$ as follows:

For $\nu \in \mathbb{N}[I]$ with $|\nu| = m$, let $R(\nu)$ denote the associative, $\Bbbk$-algebra on generators

$$1_{\boldsymbol{i}} \quad \text{for } \boldsymbol{i} \in \mathrm{Seq}(\nu), \tag{1.20}$$

$$x_r \quad \text{for } 1 \leqslant r \leqslant m, \tag{1.21}$$

$$\psi_r \quad \text{for } 1 \leqslant r \leqslant m-1 \tag{1.22}$$

subject to the following relations for $\boldsymbol{i}, \boldsymbol{j} \in \mathrm{Seq}(\nu)$:

$$1_{\boldsymbol{i}} 1_{\boldsymbol{j}} = \delta_{\boldsymbol{i},\boldsymbol{j}} 1_{\boldsymbol{i}}, \tag{1.23}$$

$$x_r 1_{\boldsymbol{i}} = 1_{\boldsymbol{i}} x_r, \tag{1.24}$$

$$\psi_r 1_{\boldsymbol{i}} = 1_{s_r(\boldsymbol{i})} \psi_r, \tag{1.25}$$

$$x_r x_t = x_t x_r, \tag{1.26}$$

$$\psi_r \psi_t = \psi_t \psi_r \quad \text{if } |r-t| > 1, \tag{1.27}$$

$$\psi_r \psi_r 1_{\boldsymbol{i}} = \begin{cases} 0 & \text{if } i_r = i_{r+1}, \\ 1_{\boldsymbol{i}} & \text{if } (\alpha_{i_r}, \alpha_{i_{r+1}}) = 0, \\ (x_r^{-\langle i_r, i_{r+1}\rangle} + x_{r+1}^{-\langle i_{r+1}, i_r\rangle}) 1_{\boldsymbol{i}} & \text{if } (\alpha_{i_r}, \alpha_{i_{r+1}}) \neq 0 \text{ and } i_r \neq i_{r+1}, \end{cases} \tag{1.28}$$

$$\begin{aligned} &(\psi_r \psi_{r+1} \psi_r - \psi_{r+1} \psi_r \psi_{r+1}) 1_{\boldsymbol{i}} \\ &\quad = \begin{cases} \sum_{t=0}^{-\langle i_r, i_{r+1}\rangle - 1} x_r^t x_{r+2}^{-\langle i_r, i_{r+1}\rangle - 1 - t} 1_{\boldsymbol{i}} & \text{if } i_r = i_{r+2} \text{ and } (\alpha_{i_r}, \alpha_{i_{r+1}}) \neq 0, \\ 0 & \text{otherwise}, \end{cases} \end{aligned} \tag{1.29}$$

$$(\psi_r x_t - x_{s_r(t)} \psi_r) 1_{\boldsymbol{i}} = \begin{cases} 1_{\boldsymbol{i}} & \text{if } t = r \text{ and } i_r = i_{r+1}, \\ -1_{\boldsymbol{i}} & \text{if } t = r+1 \text{ and } i_r = i_{r+1}, \\ 0 & \text{otherwise}. \end{cases} \tag{1.30}$$

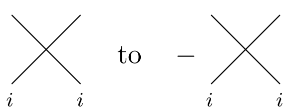
Remark 1.3. For $\boldsymbol{i}, \boldsymbol{j} \in \mathrm{Seq}(\nu)$ let ${}_{\boldsymbol{j}}S_{\boldsymbol{i}}$ be the subset of S_m consisting of permutations w that take $\boldsymbol{i}$ to $\boldsymbol{j}$ via the standard action of permutations on sequences, defined above. Denote the subset $\{\widehat{w}\}_{w \in {}_{\boldsymbol{j}}S_{\boldsymbol{i}}}$ of $1_{\boldsymbol{j}} R 1_{\boldsymbol{i}}$ by ${}_{\boldsymbol{j}}\widehat{S}_{\boldsymbol{i}}$. It was shown in [31,33] that the vector space $1_{\boldsymbol{j}} R(\nu) 1_{\boldsymbol{i}}$ has a basis consisting of elements of the form

$$\left\{ \psi_{\widehat{w}} \cdot x_1^{a_1} \dots x_m^{a_m} 1_{\boldsymbol{i}} \;\middle|\; \widehat{w} \in {}_{\boldsymbol{j}}\widehat{S}_{\boldsymbol{i}},\ a_r \in \mathbb{Z}_{\geqslant 0} \right\}. \tag{1.31}$$

Rouquier has defined a generalization of the algebras R, where the relations depend on Hermitian matrices [53]. The results of this paper will extend to these algebras whenever the Hermitian matrices give rise to graded algebras R.

1.1.4. The involution σ

Flipping a diagram about a vertical axis and simultaneously taking



(in other words, multiplying the diagram by $(-1)^s$ where s is the number of times equally labelled strands intersect) is an involution $\sigma = \sigma_\nu$ of $R(\nu)$. Let w_0 denote the longest element of $S_{|\nu|}$. We can specify σ algebraically as follows:

$$\begin{aligned} \sigma : R(\nu) &\to R(\nu), \\ 1_{\boldsymbol{i}} &\mapsto 1_{w_0(\boldsymbol{i})}, \\ x_r &\mapsto x_{|\nu|+1-r}, \\ \psi_r 1_{\boldsymbol{i}} &\mapsto (-1)^{\delta_{i_r i_{r+1}}} \psi_{|\nu|-r} 1_{w_0(\boldsymbol{i})}. \end{aligned} \tag{1.32}$$

Given an $R(\nu)$-module M, we let $\sigma^* M$ denote the $R(\nu)$-module whose underlying set is M but with twisted action $r \cdot u = \sigma(r)u$.

1.1.5. Graded characters

Define the graded character $\mathrm{ch}(M)$ of a graded finitely-generated $R(\nu)$-module M as

$$\mathrm{ch}(M) = \sum_{\boldsymbol{i} \in \mathrm{Seq}(\nu)} \mathrm{gdim}(1_{\boldsymbol{i}} M) \cdot \boldsymbol{i}.$$

The character is an element of the free $\mathbb{Z}((q))$-module with the basis $\mathrm{Seq}(\nu)$; when M is finite dimensional, $\mathrm{ch}(M)$ is an element of the free $\mathbb{Z}[q, q^{-1}]$-module with basis $\mathrm{Seq}(\nu)$.

2. Functors on the module category

2.1. Categories of graded modules

We form the direct sum

$$R = \bigoplus_{\nu \in \mathbb{N}[I]} R(\nu).$$

This is a non-unital ring. However, R is an idempotented ring with the elements $1_\nu \in R(\nu)$ giving a system of mutually orthogonal idempotents. Observe that the appropriate notion of unital module M for idempotented rings is the requirement that $M = \bigoplus_{\nu \in \mathbb{N}[I]} 1_\nu M$.

Let $R(\nu)$-mod be the category of finitely-generated graded left $R(\nu)$-modules, $R(\nu)$-fmod be the category of finite dimensional graded $R(\nu)$-modules, and $R(\nu)$-pmod be the category of projective objects in $R(\nu)$-mod. The morphisms in each of these three categories are grading-preserving module homomorphisms.

By various categories of R-modules we will mean direct sums of corresponding categories of $R(\nu)$-modules:

$$R\text{-mod} \stackrel{\text{def}}{=} \bigoplus_{\nu \in \mathbb{N}[I]} R(\nu)\text{-mod},$$

$$R\text{-fmod} \stackrel{\text{def}}{=} \bigoplus_{\nu \in \mathbb{N}[I]} R(\nu)\text{-fmod},$$

$$R\text{-pmod} \stackrel{\text{def}}{=} \bigoplus_{\nu \in \mathbb{N}[I]} R(\nu)\text{-pmod}.$$

By a simple $R(\nu)$-module we mean a simple object in the category $R(\nu)$-mod. In this paper we will be primarily concerned with the category of finite dimensional $R(\nu)$-modules. Note that this category contains all of the simples. Henceforth, by an $R(\nu)$-module we will mean a finite dimensional graded $R(\nu)$-module, unless we say otherwise. We will denote the zero module by $\mathbf{0}$.

For any two $R(\nu)$-modules M, N denote by $\mathrm{Hom}(M,N)$ or $\mathrm{Hom}_{R(\nu)}(M,N)$ the $\Bbbk$-vector space of degree preserving homomorphisms, and by $\mathrm{Hom}(M\{r\},N) = \mathrm{Hom}(M,N\{-r\})$ the space of homogeneous homomorphisms of degree r. Here $N\{r\}$ denotes N with the grading shifted up by r, so that $\mathrm{ch}(N\{r\}) = q^r \mathrm{ch}(N)$. Then we write

$$\mathrm{HOM}(M,N) := \bigoplus_{r\in\mathbb{Z}} \mathrm{Hom}\bigl(M, N\{r\}\bigr), \tag{2.1}$$

for the $\mathbb{Z}$-graded $\Bbbk$-vector space of all $R(\nu)$-module morphisms.

Though it is essential to work with the degree preserving morphisms to get the $\mathbb{Z}[q,q^{-1}]$-module structure for the categorification theorems in [31,33], for our purposes it will often be convenient to work with degree homogeneous morphisms, but not necessarily degree preserving, in the various categories of graded modules introduced above. Since any homogeneous morphism can be interpreted as a degree preserving morphism by shifting the grading on the source or target, all results stated using homogeneous morphisms can be recast as degree zero morphisms for an appropriate shift on the source or target. For this reason, throughout the paper we define $M \cong N$ to mean there exists $r \in \mathbb{Z}$ such that M is isomorphic to $N\{r\}$ as graded modules, and all isomorphisms will implicitly mean isomorphic up to such a grading shift unless otherwise specified.

2.2. *Induction and restriction functors*

There is an inclusion of graded algebras

$$\iota_{\nu,\nu'} : R(\nu) \otimes R\bigl(\nu'\bigr) \hookrightarrow R\bigl(\nu+\nu'\bigr)$$

given graphically by putting the diagrams next to each other. It takes the idempotent $1_{\boldsymbol{i}} \otimes 1_{\boldsymbol{j}}$ to $1_{\boldsymbol{i}\boldsymbol{j}}$ and the unit element $1_\nu \otimes 1_{\nu'}$ to an idempotent of $R(\nu+\nu')$ denoted $1_{\nu,\nu'}$. This inclusion gives rise to restriction and induction functors denoted by $\mathrm{Res}_{\nu,\nu'}$ and $\mathrm{Ind}_{\nu,\nu'}$, respectively. When it is clear from the context, or when no confusion is likely to arise, we often simplify notation and write Res and Ind.

We can also consider these notions for any tuple $\underline{\nu} = (\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(k)})$ and sometimes refer to the image $R(\underline{\nu}) \stackrel{\text{def}}{=} \mathrm{Im}\,\iota_{\underline{\nu}} \subseteq R(\nu^{(1)} + \cdots + \nu^{(k)})$ as a parabolic subalgebra. This subalgebra has identity $1_{\underline{\nu}}$. Let $\mu = \nu^{(1)} + \cdots + \nu^{(k)}$, $m = \sum_r |\nu^{(r)}|$, and $P = P_{\underline{\nu}}$ be the composition $(|\nu^{(1)}|, \dots, |\nu^{(k)}|)$ of m so that S_P is the corresponding parabolic subgroup of S_m. It follows from Remark 1.3 that $R(\mu)1_{\underline{\nu}}$ is a free right $R(\underline{\nu})$-module with basis $\{\psi_{\widehat{w}} 1_{\underline{\nu}} \mid w \in S_m/S_P\}$ and $1_{\underline{\nu}} R(\mu)$ is a free left $R(\underline{\nu})$-module with basis $\{1_{\underline{\nu}} \psi_{\widehat{w}} \mid w \in S_P \backslash S_m\}$. By abuse of notation we will write S_m/S_P to denote the minimal length left coset representatives, i.e. $\{w \in S_m \mid \ell(wv) = \ell(w) + \ell(v),\ \forall v \in S_P\}$, and $S_P\backslash S_m$ for the minimal length right coset representatives.

Remark 2.1. It is easy to see that if M is an $R(\underline{\nu})$-module with basis $\mathcal{U}$ consisting of weight vectors, then $\{\psi_{\widehat{w}} \otimes u \mid u \in \mathcal{U},\ w \in S_m/S_P\}$ is a weight basis of $\mathrm{Ind}_{\underline{\nu}} M \stackrel{\text{def}}{=} R(\mu) \otimes_{R(\underline{\nu})} M$ (where for each w we fix just one reduced expression $\widehat{w}$). Note $R(\mu) \otimes_{R(\underline{\nu})} M = R(\mu)1_{\underline{\nu}} \otimes_{R(\underline{\nu})} M$ since $\psi_{\widehat{w}} 1_{\underline{\nu}} \otimes u = \psi_{\widehat{w}} \otimes 1_{\underline{\nu}} u = \psi_{\widehat{w}} \otimes u$.

Likewise, $\mathrm{coInd}\, M \stackrel{\text{def}}{=} \mathrm{HOM}_{R(\underline{\nu})}(R(\mu), M)$, which is discussed in detail in Section 2.3 below, and has basis $\{f_{w,u} \mid u \in \mathcal{U},\ w \in S_P \backslash S_m\}$ where $f_{w,u}(h\psi_{\widehat{v}}) = hu\delta_{w,v}$ for $h \in R(\underline{\nu})$ and $v \in S_P \backslash S_m$. Note $\mathrm{Hom}_{R(\underline{\nu})}(R(\mu), M) = \mathrm{Hom}_{R(\underline{\nu})}(1_{\underline{\nu}} R(\mu), M)$ since for $f \in \mathrm{Hom}_{R(\underline{\nu})}(1_{\underline{\nu}} R(\mu), M)$, $t \in R(\mu)$, if $1_{\boldsymbol{i}} \notin R(\underline{\nu})$, i.e. $1_{\underline{\nu}} 1_{\boldsymbol{i}} = 0$, then

$$f(1_{\boldsymbol{i}} t) = 1_{\underline{\nu}} f(1_{\boldsymbol{i}} t) = f(1_{\underline{\nu}} 1_{\boldsymbol{i}} t) = f(0) = 0.$$

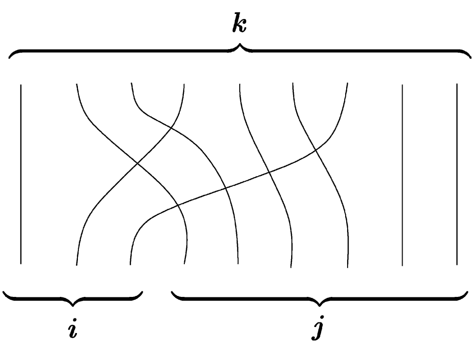
In other words, we can extend the domain of f to $R(\mu)$ by setting f to be 0 on $1_{\boldsymbol{i}} R(\mu)$ when $1_{\boldsymbol{i}} \notin R(\underline{\nu})$. Likewise any $f \in \mathrm{Hom}_{R(\underline{\nu})}(R(\mu), M)$ must be 0 on the above set.

One extremely important property of the functor $\mathrm{Ind}_{\underline{\nu}} - \stackrel{\text{def}}{=} R(\mu) \otimes_{R(\underline{\nu})} -$ is that it is left adjoint to restriction. In other words, there is a functorial isomorphism

$$\mathrm{HOM}_{R(\mu)}(\mathrm{Ind}_{\underline{\nu}} A, B) \cong \mathrm{HOM}_{R(\underline{\nu})}(A, \mathrm{Res}_{\underline{\nu}} B) \tag{2.2}$$

where A, B are finite dimensional $R(\underline{\nu})$- and $R(\mu)$-modules, respectively. This property is called Frobenius reciprocity and we use it repeatedly, often for deducing information about characters.

A shuffle $\boldsymbol{k}$ of a pair of sequences $\boldsymbol{i} \in \mathrm{Seq}(\nu)$, $\boldsymbol{j} \in \mathrm{Seq}(\nu')$ is a sequence together with a choice of subsequence isomorphic to $\boldsymbol{i}$ such that $\boldsymbol{j}$ is the complementary subsequence. Shuffles of $\boldsymbol{i}$, $\boldsymbol{j}$ are in a bijection with the minimal length left coset representatives of $S_{|\nu|} \times S_{|\nu'|}$ in $S_{|\nu|+|\nu'|}$. We denote by $\deg(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ the degree of the diagram in $R(\nu + \nu')$ naturally associated to the shuffle, see an example below.



When the meaning is clear, we will also denote by $\boldsymbol{k}$ the underlying sequence of the shuffle $\boldsymbol{k}$.

Given two functions f and g on sets $\mathrm{Seq}(\nu)$ and $\mathrm{Seq}(\nu')$, respectively, with values in some commutative ring which contains $\mathbb{Z}[q, q^{-1}]$, we define their (quantum) shuffle product $f \sqcup\!\sqcup g$ (see [42] and references therein) as the function on $\mathrm{Seq}(\nu + \nu')$ given by

$$(f \sqcup\!\sqcup g)(\boldsymbol{k}) = \sum_{\boldsymbol{i},\boldsymbol{j}} q^{\deg(\boldsymbol{i},\boldsymbol{j},\boldsymbol{k})} f(\boldsymbol{i}) g(\boldsymbol{j}),$$

the sum is over all ways to represent $\boldsymbol{k}$ as a shuffle of $\boldsymbol{i}$ and $\boldsymbol{j}$. Given $M \in R(\nu)$-mod and $N \in R(\nu')$-mod we construct the $R(\nu) \otimes R(\nu')$-module denoted by $M \boxtimes N$ in the obvious way. It was shown in [31] that

$$\mathrm{ch}\big(\mathrm{Ind}_{\nu,\nu'}(M \boxtimes N)\big) = \mathrm{ch}(M) \sqcup\!\sqcup\, \mathrm{ch}(N).$$

A similar statement holds for characters of induced $R(\underline{\nu})$-modules by the transitivity of induction. This statement can be seen as a special case of the Mackey formula which describes a filtration on the restriction of an induced module (from one parabolic to another).

More precisely, in the case of maximal parabolics, the Mackey formula says the graded $(R(\nu) \otimes R(\nu'), R(\nu'') \otimes R(\nu'''))$-bimodule $1_{\nu,\nu'} R 1_{\nu'',\nu'''}$ has a filtration over all $\lambda \in \mathbb{N}[I]$ with subquotients isomorphic to the graded bimodules

$$\begin{aligned}&(1_\nu R 1_{\nu-\lambda,\lambda} \otimes 1_{\nu'} R 1_{\nu'+\lambda-\nu''',\nu'''-\lambda})\\ &\quad \otimes_{R'} (1_{\nu-\lambda,\nu''+\lambda-\nu} R 1_{\nu''} \otimes 1_{\lambda,\nu'''-\lambda} R 1_{\nu'''})\big\{\big(-\lambda, \nu'+\lambda-\nu'''\big)\big\},\end{aligned}$$

where $R' = R(\nu-\lambda) \otimes R(\lambda) \otimes R(\nu'+\lambda-\nu''') \otimes R(\nu'''-\lambda)$, the bilinear form $(\,,\,)$ is defined in Section 1.1.1, and such that every term above is in $\mathbb{N}[I]$. There is a natural generalization of this statement to arbitrary parabolic subalgebras.

2.3. *Co-induction*

In this section, we examine the right adjoint to restriction, the co-induction functor denoted coInd, and discuss the relationship between Ind and coInd, following the work of [58]. Using the notation of the previous section, set $\mathrm{coInd}_{R(\underline{\nu})} - := \mathrm{HOM}_{R(\underline{\nu})}(R(\mu), -)$ endowed with the module structure $(r \odot f)(t) = f(tr)$ for $r, t \in R(\mu)$, $f \in \mathrm{coInd}_{R(\underline{\nu})} -$. Now there is a functorial isomorphism

$$\mathrm{HOM}_{R(\mu)}(B, \mathrm{coInd}_{\underline{\nu}} A) \cong \mathrm{HOM}_{R(\underline{\nu})}(\mathrm{Res}_{\underline{\nu}} B, A) \tag{2.3}$$

where A, B are finite dimensional modules.

Just as w_0 denotes the longest element of S_m, let $w_P \in S_P$ denote the longest element of the parabolic subgroup, with notation as above. Let $y = w_P w_0$ in the discussion below. Note that y is a minimal length right coset representative for $S_P \backslash S_m$ and corresponds to the "longest shuffle".

Observe that for any r such that $s_r \in S_P$, $\ell(w_P s_r w_P) = 1 = \ell(w_0 s_r w_0)$ and further

$$\ell(s_r y) = 1 + \ell(y) = \ell(w_P s_r w_P y) = \ell(y w_0 s_r w_0)$$

as in fact

$$(w_P s_r w_P) y = w_P s_r w_P w_P w_0 = w_P w_0 w_0 s_r w_0 = y(w_0 s_r w_0).$$

Set

$$\sigma_{\underline{\nu}} := \sigma_{\nu^{(1)}} \otimes \sigma_{\nu^{(2)}} \otimes \cdots \otimes \sigma_{\nu^{(k)}} \tag{2.4}$$

where $\sigma_\nu : R(\nu) \to R(\nu)$ is the involution defined in Section 1.1.4.

When clear from context, let us just call $\sigma = \sigma_\mu$. Then note, $\sigma(1_{\boldsymbol{j}}) = 1_{w_0(\boldsymbol{j})}$, $\sigma(x_r) = x_{w_0(r)}$, $\sigma(\psi_r 1_{\boldsymbol{j}}) = (-1)^{\delta_{j_r j_{r+1}}} \psi_{w_0 s_r w_0} 1_{w_0(\boldsymbol{j})}$ with similar equations for $\sigma_{\underline{\nu}}$, where S_m acts on $\mathrm{Seq}(\mu)$ in the usual fashion $w(i_1, \dots, i_m) = (i_{w^{-1}(1)}, \dots, i_{w^{-1}(m)})$. In what follows, for bookkeeping purposes, we will write $u \in M$, but $\overline{u} \in \sigma^* M$ so that the σ-twisted action can be described as $r\overline{u} = \overline{\sigma(r)u}$.

Theorem 2.2.

1. *Let M be a finite dimensional $R(\underline{\nu})$-module. Then*
$$\mathrm{Ind}_{\underline{\nu}}^{\mu} M \cong \sigma_\mu^*\big(\mathrm{coInd}_{\underline{\nu}}^{\mu}\big(\sigma_{\underline{\nu}}^* M\big)\big)\big\{\deg(y)\big\}$$
as graded modules.
2. *Let A be a finite dimensional $R(\nu)$-module and B a finite dimensional $R(\eta)$-module. Then there is an isomorphism*
$$\mathrm{Ind}_{\nu,\eta}^{\nu+\eta} A \boxtimes B \cong \mathrm{coInd}_{\eta,\nu}^{\eta+\nu} B \boxtimes A.$$

Proof. We first note that statement 2 follows from a special case of assumption 1. The appropriate degree shift to make it an isomorphism of graded modules is thus $-(\eta, \nu)$. To prove assumption 1, we first construct an $R(\underline{\nu})$-module map

$$M \xrightarrow{F} \mathrm{Res}_{\underline{\nu}}^{\mu}\big(\sigma_\mu^* \,\mathrm{coInd}_{\underline{\nu}}^{\mu}\big(\sigma_{\underline{\nu}}^* M\big)\big) \tag{2.5}$$

with $\deg(F) = -\deg(y)$ and then the induced map

$$\mathrm{Ind}_{\underline{\nu}}^{\mu} M \xrightarrow{\mathcal{F}} \sigma_\mu^* \,\mathrm{coInd}_{\underline{\nu}}^{\mu}\big(\sigma_{\underline{\nu}}^* M\big) \tag{2.6}$$

also has $\deg(\mathcal{F}) = -\deg(y)$ and surjective as the image of F generates the target over $R(\mu)$. Since the two modules in question have the same dimension, they are isomorphic.

Given $u \in M$ define $f_u \in \mathrm{HOM}_{R(\underline{\nu})}(R(\mu), \sigma_{\underline{\nu}}^* M)$ by

$$f_u(\psi_{\widehat{w}}) = \overline{u}\delta_{w,y} \tag{2.7}$$

where $w \in S_P \backslash S_m$ ranges over the minimal length right coset representatives, $\widehat{w}$ is a fixed reduced expression, and $y = w_P w_0$. Observe that $\deg(f_u) = \deg(u) - \deg(y)$. We extend f_u to an $R(\underline{\nu})$-map by declaring $f_u(h\psi_{\widehat{w}}) = h f_u(\psi_{\widehat{w}})$ for $h \in R(\underline{\nu})$ which is viable by Remark 2.1. Now we define

$$\begin{aligned} F : M &\to \sigma_\mu^* \,\mathrm{coInd}_{\underline{\nu}}^{\mu}\big(\sigma_{\underline{\nu}}^* M\big), \\ u &\mapsto \overline{f_u} \end{aligned} \tag{2.8}$$

and check it is an $R(\underline{\nu})$-map. This map is homogeneous with $\deg(F) = -\deg(y)$. Note that $f_{u+u'} = f_u + f_{u'}$ so it suffices to consider only degree homogeneous weight vectors $u \in M$, i.e.

there exists $\boldsymbol{i}$ such that $1_{\boldsymbol{i}}\overline{u} = \overline{u}$ (and so $1_{w_P(\boldsymbol{i})}u = u$). In this case $f_u(1_{\boldsymbol{j}}\psi_{\widehat{w}}) = \overline{u}\delta_{w,y}\delta_{\boldsymbol{i},\boldsymbol{j}}$, and this holds regardless of whether $1_{\boldsymbol{j}} \in R(\underline{\nu})$ by Remark 2.1. In fact, by abuse of notation, we may write $1_{\boldsymbol{j}}\overline{u} = \overline{u}\delta_{\boldsymbol{i},\boldsymbol{j}}$ even when $1_{\boldsymbol{j}} \notin R(\underline{\nu})$.

The following three computations show that $F(hu) = h \odot F(u)$ for $h = 1_{\boldsymbol{j}}$, $h = x_r$ for all r, and $h = \psi_r 1_{\boldsymbol{j}}$ for r such that $s_r \in S_P$ and $\boldsymbol{j}$ such that $1_{\boldsymbol{j}} \in R(\underline{\nu})$. These computations show that F is an $R(\underline{\nu})$-map. In these computations note that with respect to $\psi_{\widehat{w}}$, by lower terms we mean elements of $\{h\psi_{\widehat{v}} \mid h \in R(\underline{\nu}),\ \ell(v) < \ell(w)\}$. From now on, assume u is a weight vector as above.

Case 1) We evaluate

$$\begin{aligned}\bigl(1_{\boldsymbol{j}}F(u)\bigr)(\psi_{\widehat{w}}) &= 1_{\boldsymbol{j}} \odot \overline{f_u}(\psi_{\widehat{w}}) = \overline{\sigma_\mu(1_{\boldsymbol{j}}) \odot f_u}(\psi_{\widehat{w}})\\ &= \overline{f_u}(\psi_{\widehat{w}}1_{w_0(\boldsymbol{j})}) = \overline{f_u}(1_{ww_0(\boldsymbol{j})}\psi_{\widehat{w}})\\ &= \overline{u}\delta_{w,y}\delta_{\boldsymbol{i},ww_0(\boldsymbol{j})} = \overline{u}\delta_{w,y}\delta_{\boldsymbol{i},yw_0(\boldsymbol{j})}\\ &= \overline{u}\delta_{w,y}\delta_{\boldsymbol{i},w_P(\boldsymbol{j})} = 1_{w_P(\boldsymbol{j})}\overline{u}\delta_{w,y}\\ &= \sigma_{\underline{\nu}}(1_{\boldsymbol{j}})\overline{u}\delta_{w,y} = \overline{1_{\boldsymbol{j}}u}\delta_{w,y}\\ &= \overline{f_{1_{\boldsymbol{j}}u}}(\psi_{\widehat{w}}) = F(1_{\boldsymbol{j}}u)(\psi_{\widehat{w}})\end{aligned} \tag{2.9}$$

so that $1_{\boldsymbol{j}}F(u) = F(1_{\boldsymbol{j}}u)$.

Case 2) We compute

$$\begin{aligned}\bigl(x_rF(u)\bigr)(\psi_{\widehat{w}}) &= (x_r \odot \overline{f_u})(\psi_{\widehat{w}}) = \overline{\sigma_\mu(x_r) \odot f_u}(\psi_{\widehat{w}})\\ &= \overline{f_u}(\psi_{\widehat{w}}x_{w_0(r)})\\ &= \overline{f_u}(x_{ww_0(r)}\psi_{\widehat{w}} + \text{lower terms})\\ &= \begin{cases}\overline{f_u}(x_{w_P(r)}\psi_{\widehat{y}}) & \text{if } w = y\\ 0 & \text{else}\end{cases}\\ &= \begin{cases}x_{w_P(r)}\overline{u} & \text{if } w = y\\ 0 & \text{else}\end{cases}\\ &= \begin{cases}\overline{x_ru} & \text{if } w = y\\ 0 & \text{else}\end{cases}\\ &= \overline{f_{x_ru}}(\psi_{\widehat{w}}) = F(x_ru)(\psi_{\widehat{w}})\end{aligned} \tag{2.10}$$

so that $F(x_ru) = x_rF(u)$ for any r.

Case 3) Let r be such that $s_r \in S_P$, and $\boldsymbol{j}$ be such that $\psi_r 1_{\boldsymbol{j}} \in R(\underline{\nu})$. Recall that then $w_P s_r w_P \in S_P$ as well, and furthermore $\sigma_{\underline{\nu}}(\psi_r 1_{\boldsymbol{j}}) = \psi_{w_P s_r w_P} 1_{w_P(\boldsymbol{j})} \in R(\underline{\nu})$. We compute

$$\begin{aligned}\psi_r 1_{\boldsymbol{j}}F(u)(\psi_{\widehat{w}}) &= (\psi_r 1_{\boldsymbol{j}} \odot \overline{f_u})(\psi_{\widehat{w}})\\ &= \overline{f_u}\bigl(\psi_{\widehat{w}}\sigma_\mu(\psi_r 1_{\boldsymbol{j}})\bigr) = \overline{f_u}\bigl(\psi_{\widehat{w}}(-1)^{\delta_{j_r,j_{r+1}}}\psi_{w_0s_rw_0}1_{w_0(\boldsymbol{j})}\bigr)\\ &= \begin{cases}(-1)^{\delta_{j_r,j_{r+1}}}\overline{f_u}((\psi_{w_Ps_rw_P}\psi_{\widehat{y}} + \text{lower terms})1_{w_0(\boldsymbol{j})}) & \text{if } w = y\\ (-1)^{\delta_{j_r,j_{r+1}}}\overline{f_u}((\text{lower terms})1_{w_0(\boldsymbol{j})}) & \text{if } w \neq y\end{cases}\end{aligned}$$

$$
\begin{aligned}
&= \begin{cases} (-1)^{\delta_{j_r,j_{r+1}}} \overline{f_u}(\psi_{w_P s_r w_P} 1_{y w_0(\boldsymbol{j})} \psi_{\widehat{y}}) & \text{if } w = y \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} (-1)^{\delta_{j_r,j_{r+1}}} \psi_{w_P s_r w_P} 1_{w_P(\boldsymbol{j})} \overline{f_u}(\psi_{\widehat{y}}) & \text{if } w = y \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} \sigma_{\underline{\nu}}(\psi_r 1_{\boldsymbol{j}})\overline{u} & \text{if } w = y \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} \overline{\psi_r 1_{\boldsymbol{j}} u} & \text{if } w = y \\ 0 & \text{else} \end{cases} \\
&= \overline{f_{\psi_r 1_{\boldsymbol{j}} u}}(\psi_{\widehat{w}}) \\
&= F(\psi_r 1_{\boldsymbol{j}} u)(\psi_{\widehat{w}}),
\end{aligned} \tag{2.11}
$$

so that $\psi_r 1_{\boldsymbol{j}} F(u) = F(\psi_r 1_{\boldsymbol{j}} u)$.

Note the image of F contains all of the $\overline{f_u}$ as u ranges over a weight basis of M. Hence the image of $\mathcal{F} : \mathrm{Ind}_{\underline{\nu}}^{\mu} M \to \sigma_{\mu}^* \mathrm{coInd}_{\underline{\nu}}^{\mu}(\sigma_{\underline{\nu}}^* M)$ contains all of the $h \odot \overline{f_u}$ for $h \in R(\mu)$. We shall argue this contains a basis of $\sigma_{\mu}^* \mathrm{coInd}_{\underline{\nu}}^{\mu} \sigma_{\underline{\nu}}^* M$ which will show that $\mathcal{F}$ is surjective. Recall from Remark 2.1 that $\sigma_{\mu}^* \mathrm{coInd}_{\underline{\nu}}^{\mu}(\sigma_{\underline{\nu}}^* M)$ has a basis of "bump functions" of the form $\overline{f_{w,u}}$ and in this notation $\overline{f_u} = \overline{f_{y,u}}$. As in [58], we can show the $\psi_{\widehat{v}} \odot \overline{f_{y,u}}$ for appropriate v are triangular with respect to the $\{\overline{f_{w,u'}}\}$ so contain a basis. Since the dimensions of the induced and co-induced modules are the same, $\mathcal{F}$ is in fact an isomorphism. □

2.4. Simple $R(mi)$-modules

Simple modules for the algebra $R(mi)$ play a key role in this paper. There are several constructions of these modules.

Throughout this section let $\boldsymbol{i} = i^m$. Consider the graded algebra $\Bbbk[x_{1,\boldsymbol{i}}, \dots, x_{m,\boldsymbol{i}}]$ with $\deg(x_{t,\boldsymbol{i}}) = (\alpha_i, \alpha_i)$. Up to isomorphism and grading shift, there is a unique graded irreducible module $L(i^m)$ for the ring $R(mi)$ given as the quotient of $\Bbbk[x_{1,\boldsymbol{i}}, \dots, x_{m,\boldsymbol{i}}]$ by the ideal generated by homogeneous symmetric polynomials with positive degree, see [31, Section 2.2]. This module can alternatively be described as the induced module from the trivial R'-module, where R' is the subalgebra of $R(mi)$ generated by $\psi_{1,\boldsymbol{i}}, \dots, \psi_{m-1,\boldsymbol{i}}$ and symmetric polynomials in $\Bbbk[x_{1,\boldsymbol{i}}, \dots, x_{m,\boldsymbol{i}}]$. Note the trivial R'-module is its unique one-dimensional module, on which all $\psi_{r,\boldsymbol{i}}$ and $\sum_{r=1}^{m} x_{r,\boldsymbol{i}}^k$ act as 0, where $1 \leqslant r \leqslant m$ and $k \geqslant 1$.

Furthermore, this irreducible module $L(i^m)$ is isomorphic to the module induced from the one-dimensional graded module $L = L(i) \boxtimes \cdots \boxtimes L(i)$ over $\Bbbk[x_{1,\boldsymbol{i}}, \dots, x_{m,\boldsymbol{i}}]$ on which $x_{1,\boldsymbol{i}}, \dots, x_{m,\boldsymbol{i}}$ all act trivially. In this paper we fix the grading shift on this unique simple module $L(i^m)\{r\}$ so that

$$
\mathrm{ch}\big(L\big(i^m\big)\big) = [m]_i^! i^m. \tag{2.12}
$$

In [20, Proposition 2.8], it is not only shown that for any $u \in L(i^m)$, $1 \leqslant r \leqslant m$, and $k \geqslant m$ that $x_r^k u = 0$, but also that there exists $\widetilde{u} \in L(i^m)$ such that $x_r^{m-1} \widetilde{u} \neq 0$ for all r.

See the third statement in Section 2.5.1 for some of the important properties of $L(i^m)$, such as its behaviour under the induction and restriction functors.

2.5. Refining the restriction functor

For M in $R(\nu)$-mod and $i \in I$ let

$$\Delta_i M = (1_{\nu-i} \otimes 1_i)M = \mathrm{Res}_{\nu-i,i}\, M,$$

and, more generally,

$$\Delta_{i^n} M = (1_{\nu-ni} \otimes 1_{ni})M = \mathrm{Res}_{\nu-ni,ni}\, M.$$

We view Δ_{i^n} as a functor into the category $R(\nu-ni) \otimes R(ni)$-mod. By Frobenius reciprocity, there are functorial isomorphisms

$$\mathrm{HOM}_{R(\nu)}\big(\mathrm{Ind}_{\nu-ni,ni}\, N \boxtimes L\big(i^n\big), M\big) \cong \mathrm{HOM}_{R(\nu-ni)\otimes R(ni)}\big(N \boxtimes L\big(i^n\big), \Delta_{i^n} M\big), \tag{2.13}$$

for M as above and $N \in R(\nu-ni)$-mod.

Define

$$e_i := \mathrm{Res}_{\nu-i}^{\nu-i,i} \circ \Delta_i : R(\nu)\text{-fmod} \to R(\nu-i)\text{-fmod} \tag{2.14}$$

and for $M \in R(\nu)$-fmod, set

$$\widetilde{e}_i M := \mathrm{soc}\, e_i M, \tag{2.15}$$

$$\widetilde{f}_i M := \mathrm{cosoc}\, \mathrm{Ind}_{\nu,i}^{\nu+i}\, M \boxtimes L(i), \tag{2.16}$$

$$\varepsilon_i(M) := \max\big\{n \geqslant 0 \bigm| \widetilde{e}_i^{\,n} M \neq \mathbf{0}\big\}. \tag{2.17}$$

We also define their so-called σ-symmetric versions, which are indicated with a $\vee$. Note that $\sigma^*(\Delta_i(\sigma^* M)) = \mathrm{Res}_{i,\nu-i}\, M$. Set

$$e_i^\vee := \mathrm{Res}_{\nu-i}^{i,\nu-i} \circ \mathrm{Res}_{i,\nu-i} : R(\nu)\text{-fmod} \to R(\nu-i)\text{-fmod}, \tag{2.18}$$

$$\widetilde{e}_i^{\,\vee} M := \sigma^*\big(\widetilde{e}_i\big(\sigma^* M\big)\big) = \mathrm{soc}\, e_i^\vee M, \tag{2.19}$$

$$\widetilde{f}_i^{\,\vee} M := \sigma^*\big(\widetilde{f}_i\big(\sigma^* M\big)\big) = \mathrm{cosoc}\, \mathrm{Ind}_{i,\nu}^{\nu+i}\, L(i) \boxtimes M, \tag{2.20}$$

$$\varepsilon_i^\vee(M) := \varepsilon_i\big(\sigma^* M\big) = \max\big\{m \geqslant 0 \bigm| \big(\widetilde{e}_i^{\,\vee}\big)^m M \neq \mathbf{0}\big\}. \tag{2.21}$$

Observe that the functors e_i and $e_i^\vee$ are exact. Although the functors $\widetilde{e}_i$ and $\widetilde{f}_i$ can be defined on any module, in this paper we will only apply them to simple modules. It is a theorem of [31] that if M is irreducible, so are $\widetilde{f}_i M$ and $\widetilde{e}_i M$ (as long as the latter is nonzero), and likewise for $\widetilde{f}_i^{\,\vee} M$ and $\widetilde{e}_i^{\,\vee} M$. This is stated below along with other key properties.

2.5.1. Properties of the functors $\widetilde{e}_i$ and $\widetilde{f}_i$ on simple modules

In this section we give a long list of results that were proven in [31] about simple $R(\nu)$-modules and their behaviour under induction and restriction. They extend to the symmetrizable case by the results in [33]. We will use them freely throughout the paper.

1.
$$\mathrm{ch}(\Delta_{i^n} M) = \sum_{\boldsymbol{j} \in \mathrm{Seq}(\nu - ni)} \mathrm{gdim}(1_{\boldsymbol{j} i^n} M) \cdot \boldsymbol{j},$$
where we view $\Delta_{i^n} M$ as a module over the subalgebra $R(\nu - ni)$ of $R(\nu - ni) \otimes R(ni)$.
2. Let $N \in R(\nu)$-mod be irreducible and $M = \mathrm{Ind}_{\nu,ni} N \boxtimes L(i^n)$. Let $\varepsilon = \varepsilon_i(N)$.
   (a) $\Delta_{i^{\varepsilon+n}} M \cong \widetilde{e}_i^{\,\varepsilon} N \boxtimes L(i^{\varepsilon+n})$.
   (b) $\mathrm{cosoc}\, M$ is irreducible, and $\mathrm{cosoc}\, M \cong \widetilde{f}_i^{\,n} N$, $\Delta_{i^{\varepsilon+n}} \widetilde{f}_i^{\,n} N \cong \widetilde{e}_i^{\,\varepsilon} N \boxtimes L(i^{\varepsilon+n})$, and $\varepsilon_i(\widetilde{f}_i^{\,n} N) = \varepsilon + n$.
   (c) All other composition factors L of M have $\varepsilon_i(L) < \varepsilon + n$.
   (d) $\widetilde{f}_i^{\,n} N$ occurs with multiplicity one as a composition factor of M.
3. Let $\underline{\mu} = (i^{\mu_1}, \dots, i^{\mu_r})$ with $\sum_{k=1}^r \mu_k = n$.
   (a) The module $L(i^n)$ over the algebra $R(ni)$ is the only graded irreducible module, up to isomorphism.
   (b) All composition factors of $\mathrm{Res}_{\underline{\mu}} L(i^n)$ are isomorphic to $L(i^{\mu_1}) \boxtimes \cdots \boxtimes L(i^{\mu_r})$, and $\mathrm{soc}(\mathrm{Res}_{\underline{\mu}} L(i^n))$ is irreducible.
   (c) $\widetilde{e}_i L(i^n) \cong L(i^{n-1})$.
4. Let $M \in R(\nu)$-mod be irreducible with $\varepsilon_i(M) > 0$. Then $\widetilde{e}_i M = \mathrm{soc}(e_i M)$ is irreducible and $\varepsilon_i(\widetilde{e}_i M) = \varepsilon_i(M) - 1$. Socles of $e_i M$ are pairwise non-isomorphic for different $i \in I$.
5. For irreducible $M \in R(\nu)$-mod let $m = \varepsilon_i(M)$. Then the socle of $e_i^m M$ is isomorphic to $\widetilde{e}_i^m M^{\oplus [m]_i^!}$.
6. For irreducible modules $M \in R(\nu)$-mod and $N \in R(\nu + i)$-mod we have $\widetilde{f}_i M \cong N$ if and only if $\widetilde{e}_i N \cong M$.
7. Let $M, N \in R(\nu)$-mod be irreducible. Then $\widetilde{f}_i M \cong \widetilde{f}_i N$ if and only if $M \cong N$. Assuming $\varepsilon_i(M), \varepsilon_i(N) > 0$, $\widetilde{e}_i M \cong \widetilde{e}_i N$ if and only if $M \cong N$.

2.6. The algebras $R^\Lambda(\nu)$

For $\Lambda = \sum_{i \in I} \lambda_i \Lambda_i \in P^+$ consider the two-sided ideal $\mathcal{J}_\nu^\Lambda$ of $R(\nu)$ generated by elements $x_{1,\boldsymbol{i}}^{\lambda_{i_1}}$ over all sequences $\boldsymbol{i} \in \mathrm{Seq}(\nu)$. We sometimes write $\mathcal{J}_\nu^\Lambda = \mathcal{J}^\Lambda$ when no confusion is likely to arise. Define

$$R^\Lambda(\nu) := R(\nu)/\mathcal{J}_\nu^\Lambda. \tag{2.22}$$

By analogy with the Ariki–Koike cyclotomic quotient of the affine Hecke algebra [5] (see also [3]) this algebra is called the cyclotomic quotient at weight Λ of $R(\nu)$. As above we form the non-unital ring

$$R^\Lambda = \bigoplus_{\nu \in \mathbb{N}[I]} R^\Lambda(\nu). \tag{2.23}$$

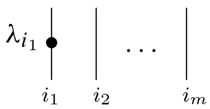
In type A the following proposition is essentially contained in [9, Section 2.2]. Here we give the natural extension to arbitrary type.

Proposition 2.3.

1. *For all* $\boldsymbol{i} \in \mathrm{Seq}(\nu)$ *and any* $\Lambda \in P^+$ *the elements* $x_{r,\boldsymbol{i}}$ *are nilpotent for all* $1 \leqslant r \leqslant |\nu|$.
2. *The algebra* $R^\Lambda(\nu)$ *is finite dimensional.*

Proof. This is left as an exercise for the reader, see [9]. $\square$

In terms of the graphical calculus the cyclotomic quotient $R^\Lambda(\nu)$ is the quotient of $R(\nu)$ by the ideal generated by



$$= \quad 0 \tag{2.24}$$

over all sequences $\boldsymbol{i}$ in $\mathrm{Seq}(\nu)$.

For bookkeeping purposes we will denote $R^\Lambda(\nu)$-modules in calligraphic font $\mathcal{M}$ but $R(\nu)$-modules by M.

We introduce functors

$$\mathrm{infl}_\Lambda : R^\Lambda(\nu)\text{-mod} \to R(\nu)\text{-fmod}, \qquad \mathrm{pr}_\Lambda : R(\nu)\text{-fmod} \to R^\Lambda(\nu)\text{-mod} \tag{2.25}$$

where infl_Λ is the inflation along the epimorphism $R(\nu) \to R^\Lambda(\nu)$, so that $\mathcal{M} = \mathrm{infl}_\Lambda \mathcal{M}$ on the level of sets. If $\mathcal{M}, \mathcal{N}$ are $R^\Lambda(\nu)$-modules, then

$$\mathrm{Hom}_{R^\Lambda(\nu)}(\mathcal{M}, \mathcal{N}) \cong \mathrm{Hom}_{R(\nu)}(\mathrm{infl}_\Lambda \mathcal{M}, \mathrm{infl}_\Lambda \mathcal{N}).$$

Note $\mathcal{M}$ is irreducible if and only if $\mathrm{infl}_\Lambda \mathcal{M}$ is. We define $\mathrm{pr}_\Lambda M = M/\mathcal{J}^\Lambda M$. If M is irreducible then $\mathrm{pr}_\Lambda M$ is either irreducible or zero. Observe infl_Λ is an exact functor and its left adjoint is pr_Λ which is only right exact.

Proposition 2.4. *Let* $\Lambda = \sum_{i \in I} \lambda_i \Lambda_i \in P^+$ *and let* M *be a simple* $R(\nu)$*-module. Then*

1. $\mathcal{J}^\Lambda M = \mathbf{0}$ *iff* $\mathrm{pr}_\Lambda M \neq \mathbf{0}$ *iff* $\varepsilon_i^\vee(M) \leqslant \lambda_i$ *for all* $i \in I$. *When these conditions hold, we may identify* M *with the* $R^\Lambda(\nu)$*-module* $\mathrm{pr}_\Lambda M$.
2. $\mathcal{J}^\Lambda M = M$ *if and only if there exists some* $i \in I$ *such that* $\varepsilon_i^\vee(M) > \lambda_i$.

We omit the proof of the above proposition. It follows from a careful study of the simple module $L(i^m)$, as in [31, Lemma 2.1] combined with the properties listed in part 2 of Section 2.5.1. The second statement follows from the first as M is simple. It also follows that when Λ is large enough $\mathcal{J}^\Lambda M = \mathbf{0}$, and such Λ always exist. Since any simple M is finite dimensional, it suffices to take $\lambda_i > \dim_{\Bbbk} M$ to ensure Λ is large enough.

Let $\mathcal{M}$ be an irreducible $R^\Lambda(\nu)$-module. As in Section 2.5 define

$$\begin{aligned} e_i^\Lambda \mathcal{M} &= \mathrm{pr}_\Lambda \circ e_i \circ \mathrm{infl}_\Lambda \mathcal{M} : R^\Lambda(\nu)\text{-mod} \to R^\Lambda(\nu - i)\text{-mod}, \\ \widetilde{e_i}^\Lambda \mathcal{M} &= \mathrm{pr}_\Lambda \circ \widetilde{e_i} \circ \mathrm{infl}_\Lambda \mathcal{M}, \\ \widetilde{f_i}^\Lambda \mathcal{M} &= \mathrm{pr}_\Lambda \circ \widetilde{f_i} \circ \mathrm{infl}_\Lambda \mathcal{M}, \\ \varepsilon_i^\Lambda(\mathcal{M}) &= \varepsilon_i(\mathrm{infl}_\Lambda \mathcal{M}). \end{aligned}$$

Let $\mathcal{M} \in R^\Lambda(\nu)$-mod and $M = \mathrm{infl}_\Lambda \mathcal{M}$. Then $\mathrm{pr}_\Lambda M = \mathcal{M}$. Since $\mathcal{J}^\Lambda M = \mathbf{0}$ then $\mathcal{J}^\Lambda e_i M = \mathbf{0}$ too, so that $e_i^\Lambda \mathcal{M}$ is an $R(\nu - i)^\Lambda$-module with $\mathrm{infl}_\Lambda(e_i^\Lambda \mathcal{M}) = e_i M$. In particular, $\dim_{\Bbbk} e_i^\Lambda \mathcal{M} = \dim_{\Bbbk} e_i M$. If furthermore $\mathcal{M}$ is irreducible, then $\widetilde{e_i}^\Lambda \mathcal{M} = \mathrm{soc}\, e_i^\Lambda \mathcal{M}$.

2.7. Ungraded modules

Write $\underline{R\text{-mod}}$, $\underline{R\text{-fmod}}$, and $\underline{R\text{-pmod}}$ for the corresponding categories of ungraded modules. There are forgetful functors

$$R\text{-mod} \to \underline{R\text{-mod}}, \qquad R\text{-fmod} \to \underline{R\text{-fmod}}, \qquad R\text{-pmod} \to \underline{R\text{-pmod}} \tag{2.26}$$

given by sending a module M to the module $\underline{M}$ obtained by forgetting the gradings, and mapping $\mathrm{HOM}(M, N)$ to $\underline{\mathrm{Hom}}(\underline{M}, \underline{N})$. Essentially not much is lost working with the ungraded modules since given an irreducible module $M \in R$-fmod, then $\underline{M}$ is irreducible in $\underline{R\text{-fmod}}$ [51, Theorem 4.4.4(v)]. Likewise, since $R^\Lambda(\nu)$ is a finite dimensional $\Bbbk$-algebra, if $K \in \underline{R^\Lambda(\nu)\text{-fmod}}$ is irreducible, then there exists an irreducible $L \in R^\Lambda(\nu)$-fmod such that $\underline{L} \cong K$. Furthermore, L is unique up to isomorphism and grading shift, see [51, Theorem 9.6.8]. Since any finite dimensional $R(\nu)$-module M can be identified with the $R^\Lambda(\nu)$-module $\mathrm{pr}_\Lambda M$ for some Λ, we also have that for any irreducible $K \in \underline{R(\nu)\text{-fmod}}$ there exists a unique, up to grading shift and isomorphism, irreducible $L \in R(\nu)$-fmod such that $\underline{L} = K$.

3. Operators on the Grothendieck group

The Grothendieck groups

$$K_0(R) = \bigoplus_{\nu \in \mathbb{N}[I]} K_0\big(R(\nu)\text{-pmod}\big), \qquad G_0(R) = \bigoplus_{\nu \in \mathbb{N}[I]} G_0\big(R(\nu)\text{-fmod}\big),$$
$$K_0\big(R^\Lambda\big) = \bigoplus_{\nu \in \mathbb{N}[I]} K_0\big(R^\Lambda(\nu)\text{-pmod}\big), \qquad G_0\big(R^\Lambda\big) = \bigoplus_{\nu \in \mathbb{N}[I]} G_0\big(R^\Lambda(\nu)\text{-fmod}\big)$$

are the direct sums of Grothendieck groups $R(\nu)$-pmod, $R(\nu)$-fmod, $R^\Lambda(\nu)$-pmod, $R^\Lambda(\nu)$-fmod respectively. The Grothendieck groups have the structure of a $\mathbb{Z}[q, q^{-1}]$-module given by shifting the grading, $q[M] = [M\{1\}]$.

The functor e_i defined in (2.14) is clearly exact so descends to an operator on the Grothendieck group

$$G_0\big(R(\nu)\text{-fmod}\big) \to G_0\big(R(\nu - i)\text{-fmod}\big) \tag{3.1}$$

and hence

$$e_i : G_0(R) \to G_0(R). \tag{3.2}$$

By abuse of notation, we will also call this operator e_i. Likewise $e_i^\Lambda : G_0(R^\Lambda) \to G_0(R^\Lambda)$. We also define divided powers

$$e_i^{(r)} : G_0(R) \to G_0(R) \tag{3.3}$$

given by $e_i^{(r)}[M] = \frac{1}{[r]_i^!}[e_i^r M]$, which are well defined by Section 2.4.

For irreducible M, we define $\widetilde{e}_i[M] = [\widetilde{e}_i M]$, $\widetilde{f}_i[M] = [\widetilde{f}_i M]$, and extend the action linearly.

The exact functors of induction and restriction induce a multiplication and comultiplication on $G_0(R)$ giving $G_0(R)$ the structure of a (twisted) bialgebra. More precisely, for $M \in R(\nu)$-fmod and $N \in R(\mu)$-fmod, the multiplication is given by $[M][N] = [\mathrm{Ind}_{\nu,\mu} M \boxtimes N]$ and the comultiplication by $\Delta[M] = \sum_{\mu_1+\mu_2=\nu} [\mathrm{Res}_{\mu_1,\mu_2} M]$. In that latter we used the fact that simple $R(\mu_1) \otimes R(\mu_2)$-modules have the form $N_1 \boxtimes N_2$ and identified $[N_1 \boxtimes N_2]$ with $[N_1] \otimes [N_2]$. There is a similar bialgebra structure on $K_0(R)$.

The main categorification results from [31,33] include the following theorem restated here for completeness. Although we do not use the results here explicitly, they are mentioned throughout the paper. The theorem below condenses those of Theorem 3.17, Propositions 3.4, 3.18 of [31] and Theorem 8 of [33].

Theorem 3.1 *(Khovanov–Lauda).*

(1) *The character map*

$$\mathrm{ch} : G_0\big(R(\nu)\text{-fmod}\big) \to \mathbb{Z}\big[q, q^{-1}\big]\mathrm{Seq}(\nu)$$

is injective.

(2) *There is an isomorphism of twisted* $\mathbb{Z}[q, q^{-1}]$*-bialgebras*

$$\gamma : {}_{\mathcal{A}}\mathbf{f} \to K_0(R) \tag{3.4}$$

such that multiplication corresponds to the exact functor Ind *and comultiplication is induced by the exact functor* Res.

Note that as a consequence of part (1) we can deduce that for any $R(\nu)$-module M its graded character $\mathrm{ch}(M)$ completely determines $[M] \in G_0(R)$.

Let us consider the maximal commutative subalgebra

$$\bigoplus_{\boldsymbol{i} \in \mathrm{Seq}(\nu)} \Bbbk[x_{1,\boldsymbol{i}}, \dots, x_{m,\boldsymbol{i}}] \subseteq R(\nu).$$

This ring was called $\mathcal{P}o\ell_\nu$ in [31]. In the notation of this paper, we could also denote it $\Bbbk[x_1, \dots, x_m]1_\nu$. Its irreducible submodules are one-dimensional, and are isomorphic to $L(i_1) \boxtimes L(i_2) \boxtimes \cdots \boxtimes L(i_m)$ and in this way correspond to $\boldsymbol{i} = (i_1, \dots, i_m) \in \mathrm{Seq}(\nu)$. In this way, we may identify $G_0(\Bbbk[x_1,\dots,x_m]1_\nu\text{-fmod})$ with $\mathbb{Z}[q,q^{-1}]\mathrm{Seq}(\nu)$. Hence one may rephrase the injectivity of the character map as saying that a module is determined by its restriction to that maximal commutative subalgebra, in their respective Grothendieck groups.

Note that the isomorphism classes of simple modules, up to grading shift, form a basis of $G_0(R)$ as a free $\mathbb{Z}[q,q^{-1}]$-module. One of the main results of this paper is that we compute the rank of $G_0(R^\Lambda(\nu)\text{-fmod})$ by realizing a crystal structure on $G_0(R^\Lambda)$ and identifying it as the highest weight crystal $B(\Lambda)$. In this language, we see the operators $\widetilde{e_i}$ and $\widetilde{f_i}$ above become crystal operators.

4. Reminders on crystals

A main result of this paper is the realization of a crystal graph structure on $G_0(R)$ which we identify as the crystal $B(\infty)$. Hence, we need to remind the reader of the language and notation of crystals. For a good introduction to crystal graphs see [28] or [21].

4.1. Monoidal category of crystals

We recall the tensor category of crystals following Kashiwara [28], see also [25,26,30].

A *crystal* is a set B together with maps

- $\mathrm{wt}: B \to P$,
- $\varepsilon_i, \varphi_i : B \to \mathbb{Z} \sqcup \{\infty\}$ for $i \in I$,
- $\widetilde{e_i}, \widetilde{f_i} : B \to B \sqcup \{0\}$ for $i \in I$,

such that

(C1) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$ for any i.
(C2) If $b \in B$ satisfies $\widetilde{e_i} b \neq 0$, then

$$\varepsilon_i(\widetilde{e_i} b) = \varepsilon_i(b) - 1, \qquad \varphi_i(\widetilde{e_i} b) = \varphi_i(b) + 1, \qquad \mathrm{wt}(\widetilde{e_i} b) = \mathrm{wt}(b) + \alpha_i. \tag{4.1}$$

(C3) If $b \in B$ satisfies $\widetilde{f_i} b \neq 0$, then

$$\varepsilon_i(\widetilde{f_i} b) = \varepsilon_i(b) + 1, \qquad \varphi_i(\widetilde{f_i} b) = \varphi_i(b) - 1, \qquad \mathrm{wt}(\widetilde{f_i} b) = \mathrm{wt}(b) - \alpha_i. \tag{4.2}$$

(C4) For $b_1, b_2 \in B$, $b_2 = \widetilde{f_i} b_1$ if and only if $b_1 = \widetilde{e_i} b_2$.
(C5) If $\varphi_i(b) = -\infty$, then $\widetilde{e_i} b = \widetilde{f_i} b = 0$.

If B_1 and B_2 are two crystals, then a *morphism* $\psi : B_1 \to B_2$ of crystals is a map

$$\psi : B_1 \sqcup \{0\} \to B_2 \sqcup \{0\}$$

satisfying the following properties:

(M1) $\psi(0)=0$.
(M2) If $\psi(b)\neq 0$ for $b\in B_1$, then

$$\mathrm{wt}\big(\psi(b)\big)=\mathrm{wt}(b),\qquad \varepsilon_i\big(\psi(b)\big)=\varepsilon_i(b),\qquad \varphi_i\big(\psi(b)\big)=\varphi_i(b). \tag{4.3}$$

(M3) For $b\in B_1$ such that $\psi(b)\neq 0$ and $\psi(\widetilde{e_i}b)\neq 0$, we have $\psi(\widetilde{e_i}b)=\widetilde{e_i}(\psi(b))$.
(M4) For $b\in B_1$ such that $\psi(b)\neq 0$ and $\psi(\widetilde{f_i}b)\neq 0$, we have $\psi(\widetilde{f_i}b)=\widetilde{f_i}(\psi(b))$.

A morphism ψ of crystals is called *strict* if

$$\psi\widetilde{e_i}=\widetilde{e_i}\psi,\qquad \psi\widetilde{f_i}=\widetilde{f_i}\psi, \tag{4.4}$$

and an *embedding* if ψ is injective.

Given two crystals B_1 and B_2 their tensor product $B_1\otimes B_2$ has underlying set $\{b_1\otimes b_2;\ b_1\in B_1 \text{ and } b_2\in B_2\}$ where we identify $b_1\otimes 0=0\otimes b_2=0$. The crystal structure is given as follows:

$$\mathrm{wt}(b_1\otimes b_2)=\mathrm{wt}(b_1)+\mathrm{wt}(b_2), \tag{4.5}$$

$$\varepsilon_i(b_1\otimes b_2)=\max\big\{\varepsilon_i(b_1),\varepsilon_i(b_2)-\big\langle h_i,\mathrm{wt}(b_1)\big\rangle\big\}, \tag{4.6}$$

$$\varphi_i(b_1\otimes b_2)=\max\big\{\varphi_i(b_1)+\big\langle h_i,\mathrm{wt}(b_2)\big\rangle,\varphi_i(b_2)\big\}, \tag{4.7}$$

$$\widetilde{e_i}(b_1\otimes b_2)=\begin{cases}\widetilde{e_i}(b_1)\otimes b_2 & \text{if } \varphi_i(b_1)\geqslant\varepsilon_i(b_2),\\ b_1\otimes\widetilde{e_i}b_2 & \text{if } \varphi_i(b_1)<\varepsilon_i(b_2),\end{cases} \tag{4.8}$$

$$\widetilde{f_i}(b_1\otimes b_2)=\begin{cases}\widetilde{f_i}(b_1)\otimes b_2 & \text{if } \varphi_i(b_1)>\varepsilon_i(b_2),\\ b_1\otimes\widetilde{f_i}b_2 & \text{if } \varphi_i(b_1)\leqslant\varepsilon_i(b_2).\end{cases} \tag{4.9}$$

Example 4.1 *(T_Λ ($\Lambda\in P$)).* Let $T_\Lambda=\{t_\Lambda\}$ with $\mathrm{wt}(t_\Lambda)=\Lambda$, $\varepsilon_i(t_\Lambda)=\varphi_i(t_\Lambda)=-\infty$, $\widetilde{e_i}t_\Lambda=\widetilde{f_i}t_\Lambda=0$. Note that the underlying set of the crystal T_Λ consists of a single node. Tensoring a crystal B with the crystal T_Λ has the effect of shifting the weight wt by Λ and leaving the other data fixed.

Example 4.2 *(B_i ($i\in I$)).* $B_i=\{b_i(n)\mid n\in\mathbb{Z}\}$ with $\mathrm{wt}(b_i(n))=n\alpha_i$,

$$\varepsilon_j\big(b_i(n)\big)=\begin{cases}-n & \text{if } i=j,\\ -\infty & \text{if } j\neq i,\end{cases}\qquad \varphi_j\big(b_i(n)\big)=\begin{cases}n & \text{if } i=j,\\ -\infty & \text{if } j\neq i,\end{cases} \tag{4.10}$$

$$\widetilde{e_j}b_i(n)=\begin{cases}b_i(n+1) & \text{if } i=j,\\ 0 & \text{if } j\neq i,\end{cases}\qquad \widetilde{f_j}b_i(n)=\begin{cases}b_i(n-1) & \text{if } i=j,\\ 0 & \text{if } j\neq i.\end{cases} \tag{4.11}$$

We write b_i for $b_i(0)$.

4.2. Description of $B(\infty)$

$B(\infty)$ is the crystal associated with the crystal graph of $\mathbf{U}_q^-(\mathfrak{g})$ where $\mathfrak{g}$ is the Kac–Moody algebra defined from the Cartan data of Section 1.1.1. One can also define $B(\infty)$ as an abstract crystal. As such, it can be characterized by Kashiwara–Saito's Proposition 4.3 below.

Proposition 4.3. *(See [30, Proposition 3.2.3].) Let B be a crystal and b_0 an element of B with weight zero. Assume the following conditions.*

(B1) $\mathrm{wt}(B) \subset Q_-$.
(B2) *b_0 is the unique element of B with weight zero.*
(B3) *$\varepsilon_i(b_0) = 0$ for every $i \in I$.*
(B4) *$\varepsilon_i(b) \in \mathbb{Z}$ for any $b \in B$ and $i \in I$.*
(B5) *For every $i \in I$, there exists a strict embedding $\Psi_i : B \to B \otimes B_i$.*
(B6) *$\Psi_i(B) \subset B \times \{\widetilde{f_i}^n b_i; n \geqslant 0\}$.*
(B7) *For any $b \in B$ such that $b \neq b_0$, there exists i such that $\Psi_i(b) = b' \otimes \widetilde{f_i}^n b_i$ with $n > 0$.*

Then B is isomorphic to $B(\infty)$.

5. Module theoretic realizations of certain crystals

5.1. The crystal $\mathcal{B}$

Let $\mathcal{B}$ denote the set of isomorphism classes of irreducible R-modules. Let $\mathbf{0}$ denote the zero module.

Let M be an irreducible $R(\nu)$-module, so that $[M] \in \mathcal{B}$. By abuse of notation, we identify M with $[M]$ in the following definitions. Hence, we are defining operators and functions on $\mathcal{B} \sqcup \{0\}$ below.

Recall from Section 2.5 the definitions

$$\widetilde{e_i} M := \operatorname{soc} e_i M, \tag{5.1}$$

$$\widetilde{f_i} M := \operatorname{cosoc} \operatorname{Ind}_{\nu,i}^{\nu+i} M \boxtimes L(i), \tag{5.2}$$

$$\varepsilon_i(M) := \max\{n \geqslant 0 \mid \widetilde{e_i}^n M \neq \mathbf{0}\} \tag{5.3}$$

and similarly the $\vee$-versions

$$\widetilde{e_i}^\vee M := \sigma^*\big(\widetilde{e_i}\big(\sigma^* M\big)\big), \tag{5.4}$$

$$\widetilde{f_i}^\vee M := \sigma^*\big(\widetilde{f_i}\big(\sigma^* M\big)\big) = \operatorname{cosoc} \operatorname{Ind}_{i,\nu}^{\nu+i} L(i) \boxtimes M, \tag{5.5}$$

$$\varepsilon_i^\vee(M) := \varepsilon_i\big(\sigma^* M\big) = \max\{m \geqslant 0 \mid \big(\widetilde{e_i}^\vee\big)^m M \neq \mathbf{0}\}. \tag{5.6}$$

For $\nu = \sum_{i \in I} \nu_i \alpha_i$, $i \in I$ and $M \in R(\nu)$-fmod set

$$\mathrm{wt}(M) = -\nu, \qquad \mathrm{wt}_i(M) = \langle h_i, \mathrm{wt}(M)\rangle. \tag{5.7}$$

Set

$$\varphi_i(M) = \varepsilon_i(M) + \langle h_i, \mathrm{wt}(M)\rangle. \tag{5.8}$$

Proposition 5.1. *The tuple $(\mathcal{B}, \varepsilon_i, \varphi_i, \widetilde{e_i}, \widetilde{f_i}, \mathrm{wt})$ defines a crystal.*

Proof. (C1) is the definition of φ_i. (C2)–(C4) were shown in [31], see Section 2.5.1. Property (C5) is vacuous as $\varphi_i(b)$ is always finite for $b \in \mathcal{B}$. $\square$

We write $\mathbb{1} \in \mathcal{B}$ for the class of the trivial $R(\nu)$-module where $\nu = \emptyset$ and $|\nu| = 0$.

One of the main theorems of this paper is Theorem 7.4 that identifies the crystal $\mathcal{B}$ as $B(\infty)$. However we need the many auxiliary results that follow before we can prove this.

5.2. The crystal $\mathcal{B} \otimes T_\Lambda$

Let M be an irreducible $R(\nu)$-module, so $M \otimes t_\Lambda \in \mathcal{B} \otimes T_\Lambda$. Then

$$\begin{aligned}
&\varepsilon_i(M \otimes t_\Lambda) = \varepsilon_i(M),\\
&\varphi_i(M \otimes t_\Lambda) = \varphi_i(M) + \lambda_i,\\
&\widetilde{e}_i(M \otimes t_\Lambda) = \widetilde{e}_i M \otimes t_\Lambda,\\
&\widetilde{f}_i(M \otimes t_\Lambda) = \widetilde{f}_i M \otimes t_\Lambda,\\
&\mathrm{wt}(M \otimes t_\Lambda) = -\nu + \Lambda.
\end{aligned}$$

5.3. The crystal $\mathcal{B}^\Lambda$

Let $\mathcal{B}^\Lambda$ denote the set of isomorphism classes of irreducible R^Λ-modules. As in the previous section, by abuse of notation we write $\mathcal{M}$ for $[\mathcal{M}]$ below. Define

$$\begin{aligned}
\widetilde{e}_i^{\,\Lambda} &: \mathcal{B}^\Lambda \to \mathcal{B}^\Lambda \sqcup \{\mathbf{0}\},\\
&\quad \mathcal{M} \mapsto \mathrm{pr}_\Lambda \circ \widetilde{e}_i \circ \mathrm{infl}_\Lambda \mathcal{M},\\
\widetilde{f}_i^{\,\Lambda} &: \mathcal{B}^\Lambda \to \mathcal{B}^\Lambda \sqcup \{\mathbf{0}\},\\
&\quad \mathcal{M} \mapsto \mathrm{pr}_\Lambda \circ \widetilde{f}_i \circ \mathrm{infl}_\Lambda \mathcal{M},\\
\varepsilon_i^\Lambda &: \mathcal{B}^\Lambda \to \mathbb{Z} \sqcup \{-\infty\},\\
&\quad \mathcal{M} \mapsto \varepsilon_i(\mathrm{infl}_\Lambda \mathcal{M}),\\
\varphi_i^\Lambda &: \mathcal{B}^\Lambda \to \mathbb{Z} \sqcup \{-\infty\},\\
&\quad \mathcal{M} \mapsto \max\bigl\{k \in \mathbb{Z} \bigm| \mathrm{pr}_\Lambda \circ \widetilde{f}_i^{\,k} \circ \mathrm{infl}_\Lambda \mathcal{M} \neq \mathbf{0}\bigr\},\\
\mathrm{wt}^\Lambda &: \mathcal{B}^\Lambda \to P,\\
&\quad \mathcal{M} \mapsto -\nu + \Lambda.
\end{aligned} \tag{5.9}$$

Note $\varepsilon_i^\Lambda(\mathcal{M}) = \max\{k \in \mathbb{Z} \mid (\widetilde{e}_i^{\,\Lambda})^k \mathcal{M} \neq \mathbf{0}\}$, and $0 \leqslant \varphi_i^\Lambda(\mathcal{M}) < \infty$.

It is true, but not at all obvious, that with this definition $\varphi_i^\Lambda(\mathcal{M}) = \varepsilon_i^\Lambda(\mathcal{M}) + \langle h_i, \mathrm{wt}^\Lambda \mathcal{M}\rangle$; see Corollary 6.22. The proof that the data $(\mathcal{B}^\Lambda, \varepsilon_i^\Lambda, \varphi_i^\Lambda, \widetilde{e}_i^{\,\Lambda}, \widetilde{f}_i^{\,\Lambda}, \mathrm{wt}^\Lambda)$ defines a crystal is delayed until Section 7.

On the level of sets define a function

$$\begin{aligned} \Upsilon : \mathcal{B}^{\Lambda} &\to \mathcal{B} \otimes T_{\Lambda}, \\ \mathcal{M} &\mapsto \operatorname{infl}_{\Lambda} \mathcal{M} \otimes t_{\Lambda}. \end{aligned} \tag{5.10}$$

The function Υ is clearly injective and satisfies

$$\varepsilon_i^{\Lambda}(\mathcal{M}) = \varepsilon_i(\Upsilon \mathcal{M}), \tag{5.11}$$

$$\Upsilon \widetilde{e_i}^{\Lambda} \mathcal{M} = \widetilde{e_i} \Upsilon \mathcal{M}, \tag{5.12}$$

$$\Upsilon \widetilde{f_i}^{\Lambda} \mathcal{M} = \begin{cases} \widetilde{f_i} \Upsilon \mathcal{M}, & \widetilde{f_i}^{\Lambda} \mathcal{M} \neq \mathbf{0}, \\ \mathbf{0}, & \widetilde{f_i}^{\Lambda} \mathcal{M} = \mathbf{0}, \end{cases} \tag{5.13}$$

$$\mathrm{wt}^{\Lambda}(\mathcal{M}) = \mathrm{wt}(\Upsilon \mathcal{M}). \tag{5.14}$$

Later we will see the relationship between $\varphi_i^{\Lambda}(\mathcal{M})$ and $\varphi_i(\operatorname{infl}_{\Lambda} \mathcal{M})$. Once this relationship is in place (see Corollary 6.22) it will imply Υ is an embedding of crystals and in particular that $\mathcal{B}^{\Lambda}$ is a crystal. In Section 7 we show that $\mathcal{B} \cong B(\infty)$ which then identifies $\mathcal{B}^{\Lambda}$ as the highest weight crystal $B(\Lambda)$.

6. Understanding $R(\nu)$-modules and the crystal data of $\mathcal{B}$

This section contains an in-depth study of simple $R(\nu)$-modules and the functor $\widetilde{f_i}$. In particular, we describe how the quantities $\varepsilon_j^{\vee}$, ε_i, φ_i^{Λ} change with the application of $\widetilde{f_j}$.

Throughout this section we assume $j \neq i$ and set $a = a_{ij} = -\langle h_i, \alpha_j \rangle$.

6.1. Jump

Given an irreducible module M, $\mathrm{pr}_{\Lambda} \widetilde{f_i} M$ is either irreducible or zero. In the following subsection, we determine exactly when the latter occurs. More specifically, we compare $\varepsilon_i^{\vee}(M)$ to $\varepsilon_i^{\vee}(\widetilde{f_i} M)$ and compute when the latter quantity "jumps" by $+1$. In this case, we show $\widetilde{f_i} M \cong \widetilde{f_i}^{\vee} M$. Understanding exactly when this jump occurs will be a key ingredient in constructing the strict embedding of crystals in Section 7.1.

One very useful byproduct of understanding co-induction is that for irreducible M if we know $\widetilde{f_i} M \cong \widetilde{f_i}^{\vee} M$ then we can easily conclude $\widetilde{f_i}^m M \cong \operatorname{Ind} M \boxtimes L(i^m) \cong \operatorname{Ind} L(i^m) \boxtimes M$, not just for $m = 1$, but for all $m \geqslant 1$, and in particular that the latter module is irreducible. We will prove this in Lemma 6.5 below. While for the main results of this paper, it suffices to understand exactly when $\widetilde{f_i} M \cong \widetilde{f_i}^{\vee} M$, we found it worthwhile to include Section 2.3 precisely for the sake of a deeper understanding of $\operatorname{Ind} M \boxtimes L(i)$.

The following proposition is a consequence of Theorem 2.2, and the properties listed in Section 2.5.1.

Proposition 6.1. *Let M be an irreducible $R(\nu)$-module. Let $n \geqslant 1$. Then*

1. $\widetilde{f_i}^n M \cong \operatorname{soc} \operatorname{coInd} M \boxtimes L(i^n) \cong \operatorname{soc} \operatorname{Ind} L(i^n) \boxtimes M$.
2. $(\widetilde{f_i}^{\vee})^n M \cong \operatorname{soc} \operatorname{coInd} L(i^n) \boxtimes M \cong \operatorname{soc} \operatorname{Ind} M \boxtimes L(i^n)$.

Proof. Let $m = \varepsilon_i(M)$ and $N = \widetilde{e}_i^{\,m} M$. Recall from Section 2.5.1

$$\mathrm{Res}_{\nu-mi,mi} M \cong N \boxtimes L\big(i^m\big). \tag{6.1}$$

We thus have a nonzero map $\mathrm{Res}_{\nu-mi,mi} M \to N \boxtimes L(i^m)$, hence a nonzero and thus injective map

$$M \to \mathrm{coInd}\, N \boxtimes L\big(i^m\big). \tag{6.2}$$

Repeating the standard arguments from [19,31] we see $M \cong \mathrm{soc}\, \mathrm{coInd}\, N \boxtimes L(i^m)$ and that all other composition factors have ε_i strictly smaller that m. Likewise we have $\widetilde{f}_i^{\,n} M \cong \mathrm{soc}\, \mathrm{coInd}\, N \boxtimes L(i^{m+n})$ and deduce statement 1, using Theorem 2.2. The proof of statement 2 is similar. □

It is necessary to understand how $\varepsilon_i^\vee$ changes with application of $\widetilde{f}_j$.

Proposition 6.2. *Let M be an irreducible $R(\nu)$-module.*

(i) *For any $i \in I$, either $\varepsilon_i^\vee(\widetilde{f}_i M) = \varepsilon_i^\vee(M)$ or $\varepsilon_i^\vee(M) + 1$.*
(ii) *For any $i, j \in I$ with $i \neq j$, we have $\varepsilon_i^\vee(\widetilde{f}_j M) = \varepsilon_i^\vee(M)$ and $\varepsilon_i(\widetilde{f}_j^{\,\vee} M) = \varepsilon_i(M)$.*

Proof. Consider $\mathrm{Ind}\, M \boxtimes L(j) \twoheadrightarrow \widetilde{f}_j M$, so by Frobenius reciprocity $\varepsilon_i^\vee(\widetilde{f}_j M) \geqslant \varepsilon_i^\vee(M)$. On the other hand, by the Shuffle Lemma

$$\varepsilon_i^\vee(\widetilde{f}_j M) \leqslant \varepsilon_i^\vee(M) + \varepsilon_i^\vee\big(L(j)\big) = \varepsilon_i^\vee(M) + \delta_{ij}. \tag{6.3}$$

In the case $i = j$ we then get $\varepsilon_i^\vee(M) \leqslant \varepsilon_i^\vee(\widetilde{f}_j M) \leqslant \varepsilon_i^\vee(M) + 1$ and in the case $i \neq j$, $\varepsilon_i^\vee(M) \leqslant \varepsilon_i^\vee(\widetilde{f}_j M) \leqslant \varepsilon_i^\vee(M)$. Applying the automorphism σ in the case $i \neq j$ also yields the symmetric statement $\varepsilon_i(\widetilde{f}_j^{\,\vee} M) = \varepsilon_i(M)$. □

Definition 6.3. Let M be an irreducible $R(\nu)$-module and let $\Lambda \in P^+$. Define

$$\varphi_i^\Lambda(M) = \max\big\{k \in \mathbb{Z} \mid \mathrm{pr}_\Lambda \widetilde{f}_i^{\,k} M \neq \mathbf{0}\big\}, \tag{6.4}$$

where we take the convention that $\widetilde{f}_i^{\,k} = \widetilde{e}_i^{\,-k}$ when $k < 0$, and that $\max \emptyset = -\infty$.

Note that $\mathrm{pr}_\Lambda M \neq \mathbf{0}$ if and only if $\varphi_i^\Lambda(M) \geqslant 0$ for all $i \in I$ by Proposition 2.4, or even for a single $i \in I$ by Proposition 6.2. Hence, by allowing φ_i^Λ to take negative values, we can use φ_i^Λ to detect which irreducible $R(\nu)$-modules are in fact $R^\Lambda(\nu)$-modules. Thus when $\varphi_i^\Lambda(M) \geqslant 0$ it agrees with $\varphi_i^\Lambda(\mathrm{pr}_\Lambda M)$ as defined in Section 5.3 which is manifestly nonnegative. By abuse of notation we call both functions φ_i^Λ.

Observe that

$$\varphi_i^\Lambda(\widetilde{f}_i M) = \varphi_i^\Lambda(M) - 1. \tag{6.5}$$

We warn the reader that with this extended definition of φ_i^Λ on $G_0(R)$, it not only takes negative values but can be equal to $-\infty$. For example, take $\Lambda = \Lambda_i$, and let $j \neq i$. Then $\widetilde{e}_i L(j) = \mathbf{0}$ and we see $\mathrm{pr}_\Lambda \widetilde{f_i}^k L(j) = \mathbf{0}$ for all $k \in \mathbb{Z}$ by Proposition 6.2. Hence $\varphi_i^\Lambda(L(j)) = -\infty$. However, this is no call for alarm, as by Proposition 2.4, we can always find a larger Λ so that $\mathrm{pr}_\Lambda M \neq \mathbf{0}$ for any given M.

Definition 6.4. Let M be a simple $R(\nu)$-module and let $i \in I$. Then

$$\mathrm{jump}_i(M) := \max\bigl\{J \geqslant 0 \bigm| \varepsilon_i^\vee(M) = \varepsilon_i^\vee\bigl(\widetilde{f_i}^J M\bigr)\bigr\}. \tag{6.6}$$

While it is clear $\mathrm{jump}_i(M) \geqslant 0$, it is less clear why $\mathrm{jump}_i(M) < \infty$. We show this in Proposition 6.7(v).

In the following lemma we collect a long list of useful characterizations of when $\mathrm{jump}_i(M) = 0$. We find it convenient to be overly thorough below and furthermore to give this lemma the name "Jump Lemma" because we use it repeatedly throughout the paper.

We remind the reader that the isomorphisms below are homogeneous but not necessarily degree preserving.

Lemma 6.5 *(Jump Lemma). Let M be irreducible. The following are equivalent:*

(1) $\mathrm{jump}_i(M) = 0$,
(2) $\widetilde{f_i} M \cong \widetilde{f_i}^\vee M$,
(3) $\widetilde{f_i}^m M \cong (\widetilde{f_i}^\vee)^m M$ *for all* $m \geqslant 1$,
(4) $\mathrm{Ind}\, M \boxtimes L(i) \cong \mathrm{Ind}\, L(i) \boxtimes M$,
(5) $\mathrm{Ind}\, M \boxtimes L(i^m) \cong \mathrm{Ind}\, L(i^m) \boxtimes M$ *for all* $m \geqslant 1$,
(6) $\widetilde{f_i} M \cong \mathrm{Ind}\, M \boxtimes L(i)$,
 (6′) $\widetilde{f_i}^\vee M \cong \mathrm{Ind}\, L(i) \boxtimes M$,
(7) $\mathrm{Ind}\, M \boxtimes L(i)$ *is irreducible,*
 (7′) $\mathrm{Ind}\, L(i) \boxtimes M$ *is irreducible,*
(8) $\mathrm{Ind}\, M \boxtimes L(i^m)$ *is irreducible for all* $m \geqslant 1$,
 (8′) $\mathrm{Ind}\, L(i^m) \boxtimes M$ *is irreducible for all* $m \geqslant 1$,
(9) $\varepsilon_i^\vee(\widetilde{f_i} M) = \varepsilon_i^\vee(M) + 1$,
 (9′) $\varepsilon_i(\widetilde{f_i}^\vee M) = \varepsilon_i(M) + 1$,
(10) $\mathrm{jump}_i(\widetilde{f_i}^m M) = 0$ *for all* $m \geqslant 0$,
(11) $\varepsilon_i^\vee(\widetilde{f_i}^m M) = \varepsilon_i^\vee(M) + m$ *for all* $m \geqslant 1$.

Proof. Pairs of "symmetric" conditions labelled by (X) and (X') are clearly equivalent to each other by applying the automorphism σ, except for (9) $\Leftrightarrow$ (9)′ which is slightly less obvious. We will show (2) $\Leftrightarrow$ (9) which then gives (2) $\Leftrightarrow$ (9)′ by σ-symmetry.

By Proposition 6.2, we have $\varepsilon_i^\vee(M) \leqslant \varepsilon_i^\vee(\widetilde{f_i} M) \leqslant \varepsilon_i^\vee(M) + 1$. This yields (1) $\Leftrightarrow$ (9). Suppose (9) holds, i.e. $\varepsilon_i^\vee(\widetilde{f_i} M) = \varepsilon_i^\vee(M) + 1 = \varepsilon_i^\vee(\widetilde{f_i}^\vee M)$. By the Shuffle Lemma,

$$\mathrm{ch}\bigl(\mathrm{Ind}\, M \boxtimes L(i)\bigr)\big|_{q=1} = \mathrm{ch}\bigl(\mathrm{Ind}\, L(i) \boxtimes M\bigr)\big|_{q=1}, \tag{6.7}$$

so by the injectivity of the character map and the discussion of Section 2.7, they have the same composition factors. But $\widetilde{f_i}^\vee M$ is the unique composition factor of $\mathrm{Ind}\, L(i) \boxtimes M$ with the largest $\varepsilon_i^\vee$, forcing $\widetilde{f}_i M \cong \widetilde{f}_i^\vee M$ which yields (2). The converse of (2) $\Rightarrow$ (9) is obvious. So we have (2) $\Leftrightarrow$ (9) and by σ-symmetry also (2) $\Leftrightarrow$ (9)$'$.

Next suppose (2), i.e. $\widetilde{f}_i M \cong \widetilde{f}_i^\vee M$. This implies

$$\operatorname{cosoc} \operatorname{Ind} M \boxtimes L(i) \cong \operatorname{soc} \operatorname{coInd} L(i) \boxtimes M \cong \operatorname{soc} \operatorname{Ind} M \boxtimes L(i) \tag{6.8}$$

by Proposition 6.1. Furthermore from Section 2.5.1, $\widetilde{f}_i M$ is not only the cosocle, but occurs with multiplicity one in $\operatorname{Ind} M \boxtimes L(i)$. For it to also be the socle forces $\operatorname{Ind} M \boxtimes L(i)$ to be irreducible, yielding (7). Clearly (7) $\Leftrightarrow$ (6). Further (7) $\Rightarrow$ (4) as $\operatorname{ch}(\operatorname{Ind} M \boxtimes L(i)) = \operatorname{ch}(\operatorname{Ind} L(i) \boxtimes M)$ at $q = 1$.

Given (4) an inductive argument and transitivity of induction gives (5), that $\operatorname{Ind} M \boxtimes L(i^m) \cong \operatorname{Ind} L(i^m) \boxtimes M$ for all $m \geqslant 1$. Thus, $\widetilde{f}_i^{\,m} M \cong \operatorname{cosoc} \operatorname{Ind} M \boxtimes L(i^m) \cong \operatorname{cosoc} \operatorname{Ind} L(i^m) \boxtimes M \cong (\widetilde{f}_i^\vee)^m M$, yielding (3) and thus (11) by then evaluating $\varepsilon_i^\vee$. That (11) $\Rightarrow$ (3) is an identical argument to (9) $\Rightarrow$ (2).

Now suppose (3) holds. Again by Proposition 6.1

$$\operatorname{cosoc} \operatorname{Ind} M \boxtimes L\big(i^m\big) \cong \operatorname{soc} \operatorname{coInd} L\big(i^m\big) \boxtimes M \cong \operatorname{soc} \operatorname{Ind} M \boxtimes L\big(i^m\big) \tag{6.9}$$

so as above $\operatorname{Ind} M \boxtimes L(i^m)$ is irreducible, yielding (8), and hence it is isomorphic to $\widetilde{f}_i^{\,m} M$.

It is trivial to check (8) $\Rightarrow$ (7) $\Rightarrow$ (4) $\Rightarrow$ (2) and (6) $\Leftrightarrow$ (6)$'$, (7) $\Leftrightarrow$ (7)$'$, (8) $\Leftrightarrow$ (8)$'$. Finally, since (1) $\Leftrightarrow$ (11) we certainly have (1) $\Leftrightarrow$ (10). This completes the proof. $\square$

The following proposition gives alternate characterizations of $\operatorname{jump}_i(M)$. Although we do not prove that all five hold at this time, it is worth stating them all together now.

Proposition 6.6. *Let M be a simple $R(\nu)$-module and let $i \in I$. Then the following hold.*

(i) $\operatorname{jump}_i(M) = \min\{J \geqslant 0 \mid \widetilde{f}_i(\widetilde{f}_i^{\,J} M) \cong \widetilde{f}_i^\vee(\widetilde{f}_i^{\,J} M)\}$.
(ii) *If $\varphi_i^\Lambda(M) > -\infty$, then* $\operatorname{jump}_i(M) = \varphi_i^\Lambda(M) + \varepsilon_i^\vee(M) - \lambda_i$, *where $\Lambda = \sum_i \lambda_i \Lambda_i \in P^+$.*

Proof. We first prove (i). Let $J = \operatorname{jump}_i(M)$ and $N = \widetilde{f}_i^{\,J} M$. Then by the maximality of J, $\varepsilon_i^\vee(\widetilde{f}_i N) = \varepsilon_i^\vee(N) + 1 = \varepsilon_i^\vee(M) + 1$. By the Jump Lemma 6.5, $\widetilde{f}_i N \cong \widetilde{f}_i^\vee N$, i.e. $\widetilde{f}_i(\widetilde{f}_i^{\,J} M) \cong \widetilde{f}_i^\vee(\widetilde{f}_i^{\,J} M)$. Further, if $0 \leqslant m < J$ then

$$\varepsilon_i^\vee\big(\widetilde{f}_i^\vee \widetilde{f}_i^{\,m} M\big) = 1 + \varepsilon_i^\vee\big(\widetilde{f}_i^{\,m} M\big) = 1 + \varepsilon_i^\vee(M) > \varepsilon_i^\vee(M) = \varepsilon_i^\vee\big(\widetilde{f}_i^{\,m+1} M\big) \tag{6.10}$$

so $\widetilde{f}_i^\vee \widetilde{f}_i^{\,m} M \ncong \widetilde{f}_i \widetilde{f}_i^{\,m} M$. This yields (i).

Now we prove (ii). Again let $J = \operatorname{jump}_i(M)$. First, suppose $\varphi_i^\Lambda(M) \geqslant 0$. Then, as $\operatorname{pr}_\Lambda \widetilde{f}_i^{\,\varphi_i^\Lambda(M)} M \neq \mathbf{0}$, it follows from Propositions 6.2 and 2.4 that $\operatorname{pr}_\Lambda M \neq \mathbf{0}$. Hence $\lambda_i \geqslant \varepsilon_i^\vee(M) = \varepsilon_i^\vee(\widetilde{f}_i^{\,J} M)$. But by (11) of the Jump Lemma, $\varepsilon_i^\vee(\widetilde{f}_i^{\,J+m} M) = \varepsilon_i^\vee(M) + m$ for all $m \geqslant 0$.

Set $m = \lambda_i - \varepsilon_i^\vee(M)$. Then by the maximality of J, $\varepsilon_i^\vee(\widetilde{f}_i^{\,J+m} M) = \lambda_i$ but $\varepsilon_i^\vee(\widetilde{f}_i^{\,J+m+1} M) = \lambda_i + 1$. And by Proposition 6.2 $\varepsilon_j^\vee(\widetilde{f}_i^{\,J+m} M) = \varepsilon_j^\vee(M) \leqslant \lambda_j$. In other words $\operatorname{pr}_\Lambda \widetilde{f}_i^{\,J+m} M \neq \mathbf{0}$

but $\mathrm{pr}_\Lambda \widetilde{f}_i^{\,J+m+1} M = \mathbf{0}$, so by definition $\varphi_i^\Lambda(M) = J + m = \mathrm{jump}_i(M) + \lambda_i - \varepsilon_i^\vee(M)$. Equivalently $\mathrm{jump}_i(M) - \varphi_i^\Lambda(M) + \varepsilon_i^\vee(M) - \lambda_i$.

Second, if $-\infty < \varphi_i^\Lambda(M) < 0$, let $k = -\varphi_i^\Lambda(M)$. Note $\varepsilon_i^\vee(\widetilde{e}_i^{\,k} M) = \lambda_i$ but $\varepsilon_i^\vee(\widetilde{e}_i^{\,k-1} M) = \lambda_i + 1$ so that $\mathrm{jump}_i(\widetilde{e}_i^{\,k} M) = 0$ and hence $\mathrm{jump}_i(M) = 0$ too, by characterization (10) of the Jump Lemma. As before, $\varepsilon_i^\vee(M) = \varepsilon_i^\vee(\widetilde{f}_i^{\,k}\widetilde{e}_i^{\,k} M) = \varepsilon_i^\vee(\widetilde{e}_i^{\,k} M) + k = \lambda_i - \varphi_i^\Lambda(M)$. So again $\mathrm{jump}_i(M) = 0 = \varphi_i^\Lambda(M) + \varepsilon_i^\vee(M) - \lambda_i$. $\square$

It is clear from Proposition 6.6 that

$$\mathrm{jump}_i(\widetilde{f}_i M) = \max\bigl\{0, \mathrm{jump}_i(M) - 1\bigr\}. \tag{6.11}$$

We continue our list of characterizations of jump_i in a separate proposition below, whose proof is postponed to the end of this Section 6.4.

Proposition 6.7. *Let M be a simple $R(\nu)$-module and let $i \in I$. Then the following hold.*

(iii) $\mathrm{jump}_i(M) = \max\{J \geqslant 0 \mid \varepsilon_i(M) = \varepsilon_i((\widetilde{f}_i^{\,\vee})^J M)\}$.
(iv) $\mathrm{jump}_i(M) = \min\{J \geqslant 0 \mid \widetilde{f}_i((\widetilde{f}_i^{\,\vee})^J M) \cong \widetilde{f}_i^{\,\vee}((\widetilde{f}_i^{\,\vee})^J M)\}$.
(v) $\mathrm{jump}_i(M) = \varepsilon_i(M) + \varepsilon_i^\vee(M) + \mathrm{wt}_i(M)$.

We must delay the proof of (v) until we have proved Theorem 6.21 and consequently Corollary 6.22.

The equivalence of Proposition 6.6(i) to the definition of jump_i is σ-symmetric to the equivalence of (iii) $\Leftrightarrow$ (iv), and (i) is σ-symmetric to (iv). So once we have (v) whose right-hand side is a σ-symmetric expression, we will have all (iii)–(v) of Proposition 6.7.

Remark 6.8. Given $\Lambda, \Omega \in P^+$ and irreducible modules A and B with $\mathrm{pr}_\Lambda A \neq \mathbf{0}$, $\mathrm{pr}_\Omega A \neq \mathbf{0}$, $\mathrm{pr}_\Lambda B \neq \mathbf{0}$, $\mathrm{pr}_\Omega B \neq \mathbf{0}$, then $\varphi_i^\Lambda(A) - \varphi_i^\Lambda(B) = \varphi_i^\Omega(A) - \varphi_i^\Omega(B)$ since by Proposition 6.6(ii) we compute

$$\varphi_i^\Lambda(A) - \varphi_i^\Lambda(B) = \bigl(\mathrm{jump}_i(A) - \varepsilon_i^\vee(A) + \lambda_i\bigr) - \bigl(\mathrm{jump}_i(B) - \varepsilon_i^\vee(B) + \lambda_i\bigr) \tag{6.12}$$

$$= \mathrm{jump}_i(A) - \mathrm{jump}_i(B) + \varepsilon_i^\vee(B) - \varepsilon_i^\vee(A) \tag{6.13}$$

$$= \varphi_i^\Omega(A) - \varphi_i^\Omega(B). \tag{6.14}$$

6.2. Serre relations

In this section we discuss the quantum Serre relations (6.16) which are certain (minimal) relations that hold among the operators e_i on $G_0(R)$. We refer the reader to [33], where they prove similar relations (the vanishing of alternating sums in $K_0(R)$) hold on a certain family of projective modules in their Corollary 7. Then by the obvious generalization to the symmetrizable case of Corollary 2.15 of [31] we have

$$\sum_{r=0}^{a+1} (-1)^r e_i^{(a+1-r)} e_j e_i^{(r)} [M] = 0 \tag{6.15}$$

for all $M \in R(\nu)$-mod with $|\nu| = a+1$, where $a = -\langle h_i, \alpha_j \rangle$, and hence for all $[M] \in G_0(R)$, showing the operator

$$\sum_{r=0}^{a+1} (-1)^r e_i^{(a+1-r)} e_j e_i^{(r)} = 0. \tag{6.16}$$

Recall the divided powers $e_i^{(r)}$ are given by $e_i^{(r)}[M] = \frac{1}{[r]_i^!}[e_i^r M]$.

Furthermore, when $c \leqslant a$ the operator

$$\sum_{r=0}^{c} (-1)^r e_i^{(c-r)} e_j e_i^{(r)} \tag{6.17}$$

is never the zero operator on $G_0(R)$ by the quantum Gabber–Kac theorem [46, Theorem 33.1.3] and the work of [31,33], which essentially computes the kernel of the map from the free algebra on the generators e_i to $G_0(R)$, see Remark 1.2.

In Section 6.3.1 below, we give an alternate proof that the quantum Serre relation (6.16) holds by examining the structure of all simple $R((a+1)i+j)$-modules. We further construct simple $R(ci+j)$-modules that are witnesses to the non-vanishing of (6.17) when $c \leqslant a$. In the following remark, we give a sample argument of how understanding the simple $R(\nu)$-modules for a fixed ν gives a relation among the operators e_i on $G_0(R)$. Although we only give it in detail for a degree 2 relation among the e_i, it can be easily extended to higher degree relations.

Remark 6.9. Suppose we have explicitly constructed all simple $R(i+j)$-modules M, and have verified $(e_i e_j - e_j e_i)[M] = 0$ for all such M. (We know this is the case whenever $\langle i, j \rangle = 0$.) We will call this a degree 2 relation in the e_i's for obvious reasons. We easily see the operator $e_i e_j - e_j e_i$ is zero on $G_0(R(\mu)\text{-fmod})$ not just for $\mu = i+j$ but for any ν with $|\mu| = 0, 1, 2$. Now consider arbitrary ν with $|\nu| > 2$. Let M be any finite dimensional $R(\nu)$-module. We can write $[\mathrm{Res}_{\nu-\mu,\mu} M] = \sum_h [A_h \boxtimes B_h]$ for some simple $R(\mu)$-modules B_h with $|\mu| = 2$, or the restriction is zero. Then

$$(e_i e_j - e_j e_i)[M] = \sum_{\mu:\ |\mu|=2} \sum_h \big[A_h \boxtimes (e_i e_j - e_j e_i) B_h\big] \tag{6.18}$$

$$= \sum \sum [A_h \boxtimes \mathbf{0}] = 0. \tag{6.19}$$

Hence $e_i e_j - e_j e_i$ is zero as an operator on $G_0(R)$. However, this is a relation of the form (6.17) with $c = 0$. By the discussion above on the minimality of the quantum Serre relation, this forces $a_{ij} = 0$. Similarly, if one shows the expression (6.16) in the quantum Serre relation vanishes on all irreducible $R((a+1)i+j)$-modules, the same argument shows the relation holds on all $G_0(R)$ and that $a_{ij} \leqslant a$.

6.3. The Structure Theorems for simple $R(ci+j)$-modules

In this section we describe the structure of all simple $R(ci+j)$-modules. We will henceforth refer to Theorems 6.10, 6.11 as the Structure Theorems for simple $R(ci+j)$-modules. Throughout this section we assume $j \neq i$ and set $a = a_{ij} = -\langle h_i, \alpha_j \rangle$.

In the theorems below we introduce the notation

$$\mathcal{L}\big(i^{c-n}ji^n\big) \quad \text{and} \quad \mathcal{L}(n) \stackrel{\text{def}}{=} \mathcal{L}\big(i^{a-n}ji^n\big)$$

for the irreducible $R(ci+j)$-modules when $c \leqslant a$. They are characterized by $\varepsilon_i(\mathcal{L}(i^{c-n}ji^n)) = n$.

Theorem 6.10. *Let $c \leqslant a$ and let $\nu = ci + j$. Up to isomorphism, there exists a unique irreducible $R(\nu)$-module denoted $\mathcal{L}(i^{c-n}ji^n)$ with*

$$\varepsilon_i\big(\mathcal{L}\big(i^{c-n}ji^n\big)\big) = n \tag{6.20}$$

for each n with $0 \leqslant n \leqslant c$. Furthermore,

$$\varepsilon_i^\vee\big(\mathcal{L}\big(i^{c-n}ji^n\big)\big) = c-n \tag{6.21}$$

and

$$\mathrm{ch}\big(\mathcal{L}\big(i^{c-n}ji^n\big)\big) = [c-n]_i![n]_i!i^{c-n}ji^n. \tag{6.22}$$

In particular, in the Grothendieck group $e_i^{(c-s)}e_je_i^{(s)}[\mathcal{L}(i^{c-n}ji^n)] = 0$ unless $s = n$.

Proof. The proof is by induction on c. The case $c = 0$ is obvious; there exists a unique irreducible $R(j)$-module $L(j)$ and it obviously satisfies (6.20)–(6.22).

The case $c = 1$ is also straightforward. Since $c \leqslant a$, and so $a \neq 0$, we compute $\mathrm{Ind}\, L(i) \boxtimes L(j)$ is reducible, but has irreducible cosocle. Let

$$\mathcal{L}(ij) = \mathrm{cosoc}\, \mathrm{Ind}\, L(i) \boxtimes L(j), \tag{6.23}$$

$$\mathcal{L}(ji) = \mathrm{cosoc}\, \mathrm{Ind}\, L(j) \boxtimes L(i). \tag{6.24}$$

Note that each of the above modules is one-dimensional and satisfies (6.20)–(6.22). Observe if (6.20) did not hold for either module, then by the Jump Lemma 6.5

$$\mathrm{Ind}\, L(i) \boxtimes L(j) \cong \mathrm{Ind}\, L(j) \boxtimes L(i) \tag{6.25}$$

and this module would be irreducible. Hence for all $R(i+j)$-modules M we would have

$$(e_ie_j - e_je_i)[M] = 0 \tag{6.26}$$

and in fact this relation would then hold for any ν and any irreducible $R(\nu)$-module M via Remark 6.9. But by (6.17) this would imply $a = 0$, a contradiction.

Now assume the theorem holds for some fixed $c \leqslant a$ and we will show it also holds for $c+1$ as long as $c+1 \leqslant a$. Let N be an irreducible $R((c+1)i+j)$-module with $\varepsilon_i(N) = n$.

Suppose $n > 0$. If in fact $n = 0$ consider instead $n^\vee = \varepsilon_i^\vee N$ which cannot also be 0 and perform the following argument applying the automorphism σ everywhere. Observe any other module N' such that $\varepsilon_i(N') = n$ has $\widetilde{e_i}N' \cong \widetilde{e_i}N$, forcing $N' \cong N$, which gives us the uniqueness.

Note that $\widetilde{e}_i N$ is an $R(ci+j)$-module with $\varepsilon_i(\widetilde{e}_i N) = n-1$ so by the inductive hypothesis $\widetilde{e}_i N = \mathcal{L}(i^{c+1-n} j i^{n-1})$. We have a surjection (up to grading shift)

$$\operatorname{Ind} \mathcal{L}\bigl(i^{c+1-n} j i^{n-1}\bigr) \boxtimes L(i) \twoheadrightarrow N. \tag{6.27}$$

Since $N = \operatorname{cosoc} \operatorname{Ind} \mathcal{L}(i^{c+1-n} j i^{n-1}) \boxtimes L(i)$, by Frobenius reciprocity, the Shuffle Lemma, and the fact that $L(i^m)$ is irreducible with character $[m]_i^! i^m$, either we have

$$\operatorname{ch}(N) = [c+1-n]_i^! [n]_i^! i^{c+1-n} j i^n \tag{6.28}$$

or

$$\begin{aligned} \operatorname{ch}(N) &= [c+1-n]_i^! [n]_i^! i^{c+1-n} j i^n + q^{-(\alpha_i,\alpha_j)} [c+2-n]_i^! [n-1]_i^! i^{c+2-n} j i^{n-1} && (6.29)\\ &= \operatorname{ch}\bigl(\operatorname{Ind} \mathcal{L}\bigl(i^{c+1-n} j i^{n-1}\bigr) \boxtimes L(i)\bigr). && (6.30) \end{aligned}$$

In the former case, N satisfies (6.22) and of course also (6.21). In the latter case, by the injectivity of the character map, we must have isomorphisms $N \cong \operatorname{Ind} \mathcal{L}(i^{c+1-n} j i^{n-1}) \boxtimes L(i)$ and in fact

$$\operatorname{Ind} \mathcal{L}\bigl(i^{c+1-n} j i^{n-1}\bigr) \boxtimes L(i) \cong \operatorname{Ind} L(i) \boxtimes \mathcal{L}\bigl(i^{c+1-n} j i^{n-1}\bigr). \tag{6.31}$$

Next we will show that if (6.31) holds for this n, then it holds for all $1 \leqslant n \leqslant c$.

Let $M = \operatorname{cosoc} \operatorname{Ind} L(i) \boxtimes \mathcal{L}(i^{c-n} j i^n)$ which is irreducible. By the Shuffle Lemma, either $\varepsilon_i(M) = n$ or $\varepsilon_i(M) = n+1$. If $\varepsilon_i(M) = n$, then by uniqueness part of the inductive hypothesis $\widetilde{e}_i M \cong \widetilde{e}_i N$ and so $M \cong N$. But this is impossible as $i^{c+2-n} j i^{n-1}$ can never be a constituent of $\operatorname{ch}(M)$. So we must have $\varepsilon_i(M) = n+1$. Repeating the same analysis of characters as above we must have

$$M \cong \operatorname{Ind} L(i) \boxtimes \mathcal{L}\bigl(i^{c-n} j i^n\bigr) \cong \operatorname{Ind} \mathcal{L}\bigl(i^{c-n} j i^n\bigr) \boxtimes L(i). \tag{6.32}$$

Continuing in this manner, we deduce

$$\operatorname{Ind} L(i) \boxtimes \mathcal{L}\bigl(i^{c-g} j i^g\bigr) \cong \operatorname{Ind} \mathcal{L}\bigl(i^{c-g} j i^g\bigr) \boxtimes L(i) \tag{6.33}$$

for all $n-1 \leqslant g \leqslant c$.

We may repeat the same argument applying the automorphism σ everywhere. In other words consider $\varepsilon_i^\vee(N) = c+2-n$ and start with

$$M' = \operatorname{cosoc} \operatorname{Ind} \mathcal{L}\bigl(i^{c+2-n} j i^{n-2}\bigr) \boxtimes L(i) \tag{6.34}$$

which will force $\varepsilon_i^\vee(M') = c+3-n$ and

$$\operatorname{Ind} \mathcal{L}\bigl(i^{c+2-n} j i^{n-2}\bigr) \boxtimes L(i) \cong \operatorname{Ind} L(i) \boxtimes \mathcal{L}\bigl(i^{c+2-n} j i^{n-2}\bigr). \tag{6.35}$$

Continuing as before yields isomorphisms (6.33) for $n-1 > g \geqslant 0$, in other words for all g.

Under the original assumption that the $R((c+1)i+j)$-module N does not satisfy (6.22), we have shown that every irreducible $R((c+1)i+j)$-module A satisfies

$$A \cong \operatorname{Ind} L(i) \boxtimes B \cong \operatorname{Ind} B \boxtimes L(i) \tag{6.36}$$

for some irreducible $R(ci+j)$-module B, and furthermore we have computed $\operatorname{ch}(A)$.

On closer examination of these characters, we see

$$\sum_{s=0}^{c+1} (-1)^s e_i^{(c+1-s)} e_j e_i^{(s)} [A] = 0 \tag{6.37}$$

for all such A. But then an argument similar to that in Remark 6.9 shows

$$\sum_{s=0}^{c+1} (-1)^s e_i^{(c+1-s)} e_j e_i^{(s)} [C] = 0 \tag{6.38}$$

for all irreducible $R(\nu)$-modules C for any $\nu \in \mathbb{N}[I]$. So by (6.17), (6.16) we would get $a \leqslant c$, contradicting $c+1 \leqslant a$.

So it must be that all irreducible $R((c+1)i+j)$-modules satisfy (6.20), (6.21), and (6.22). $\square$

In the previous theorem we introduced the notation $\mathcal{L}(i^{c-n} j i^n)$ for the unique simple $R(ci+j)$-module with $\varepsilon_i = n$ when $c \leqslant a$. Theorem 6.11 below extends this uniqueness to $c \geqslant a$. Recall that in the special case that $c = a$, we denote

$$\mathcal{L}(n) = \mathcal{L}\big(i^{a-n} j i^n\big).$$

The following theorem motivates why we distinguish the special case $c = a$.

Theorem 6.11. *Let* $0 \leqslant n \leqslant a$.

(i) *The module*

$$\operatorname{Ind} L\big(i^m\big) \boxtimes \mathcal{L}(n) \cong \operatorname{Ind} \mathcal{L}(n) \boxtimes L\big(i^m\big) \tag{6.39}$$

is irreducible for all $m \geqslant 0$.

(ii) *Let* $c \geqslant a$. *Let* N *be an irreducible* $R(ci+j)$-*module with* $\varepsilon_i(N) = n$. *Then* $c - a \leqslant n \leqslant c$ *and*

$$N \cong \operatorname{Ind} \mathcal{L}\big(n-(c-a)\big) \boxtimes L\big(i^{c-a}\big). \tag{6.40}$$

Proof. We first prove (6.39) for $m = 1$, from which it will follow for all m by the Jump Lemma 6.5. Let $M = \widetilde{f}_i \mathcal{L}(n) = \operatorname{cosoc} \operatorname{Ind} \mathcal{L}(n) \boxtimes L(i)$, which is irreducible. Note $\varepsilon_i(M) = n+1$ and by the Shuffle Lemma

$$e_i^{(a-n)} e_j e_i^{(n+1)} [M] \neq 0 \tag{6.41}$$

but

$$e_i^{(a+1-s)} e_j e_i^{(s)}[M] = 0 \tag{6.42}$$

unless $s = n+1$ or $s = n$. But the Serre relations (6.16) imply the following operator is identically zero:

$$\sum_{s=0}^{a+1} (-1)^s e_i^{(a+1-s)} e_j e_i^{(s)} = 0. \tag{6.43}$$

In particular,

$$\begin{aligned} 0 &= \sum_{s=0}^{a+1} (-1)^s e_i^{(a+1-s)} e_j e_i^{(s)}[M] \\ &\overset{(6.42)}{=} (-1)^n e_i^{(a+1-n)} e_j e_i^{(n)}[M] + (-1)^{n+1} e_i^{(a-n)} e_j e_i^{(n+1)}[M], \end{aligned} \tag{6.44}$$

from which we conclude, by (6.41), that

$$e_i^{(a+1-n)} e_j e_i^{(n)}[M] \neq 0. \tag{6.45}$$

This implies

$$a - n + 1 = \varepsilon_i^\vee M = \varepsilon_i^\vee \big(\widetilde{f}_i \mathcal{L}(n)\big) = \varepsilon_i^\vee \big(\mathcal{L}(n)\big) + 1 \tag{6.46}$$

so that by the Jump Lemma $\widetilde{f}_i \mathcal{L}(n) \cong \widetilde{f}_i^\vee \mathcal{L}(n)$, and consequently part (i) of the theorem also holds for all $m \geqslant 1$. (The case $m = 0$ is vacuously true.)

For part (ii), we induct on $c \geqslant a$, the case $c = a$ following directly from Theorem 6.10. Now assume the statement for general $c > a$ and consider an irreducible $R((c+1)i + j)$-module N such that $\varepsilon_i(N) = n$. If $n = 0$, then clearly $e_i^{(c+1)} e_j[N] \neq 0$ so also $e_i^{(a+1)} e_j[N] \neq 0$, which by the Serre relations (6.16) implies there exists an $n' \neq 0$ with $e_i^{(a+1-n')} e_j e_i^{(n')}[N] \neq 0$. But then $\varepsilon_i(N) \geqslant n' > 0$, which is a contradiction.

Let $M \cong \widetilde{e}_i N \neq \mathbf{0}$, so that $\varepsilon_i(M) = n - 1$ and by the inductive hypothesis

$$M \cong \operatorname{Ind} \mathcal{L}\big(n - 1 - (c - a)\big) \boxtimes L\big(i^{c-a}\big).$$

Hence, by part (i) and the Jump Lemma

$$N \cong \widetilde{f}_i M \cong \operatorname{Ind} \mathcal{L}\big(n - \big((c+1) - a\big)\big) \boxtimes L\big(i^{c+1-a}\big). \tag{6.47}$$

Consequently $n \geqslant c + 1 - a$. As N is an irreducible $R((c+1)i + j)$-module, clearly $c + 1 \geqslant n$. $\square$

Observe that from Theorems 6.10, 6.11 and the Shuffle Lemma, we have computed the character (up to grading shift) of all irreducible $R(ci + j)$-modules.

6.3.1. The generators and relations proof

In this section, we give alternative proofs of the Structure Theorems 6.10 and 6.11 using the description of $R(\nu)$ via generators and relations. In particular, we do not use the Serre relations (6.16) and in fact one could instead deduce that the Serre relations hold from these theorems.

We first set up some useful notation. For this section let

$$\boldsymbol{i}(b,c) = \underbrace{i\dots i}_{b}\, j\, \underbrace{i\dots i}_{c}.$$

Let $\{u_r \mid 1 \leqslant r \leqslant m!\}$ be a (weight) basis of $L(i^m)$, $\{y_s \mid 1 \leqslant s \leqslant n!\}$ be a basis of $L(i^n)$, and $\{v\}$ be a basis of $L(j)$. Recall the following fact about the irreducible module $L(i^m)$. For any $u \in L(i^m)$

$$x_r^k u = 0, \tag{6.48}$$

for all $k \geqslant m$, and $1 \leqslant r \leqslant m$. Further if $u \neq 0$ then $L(i^m) = R(mi)u$, and $1_{\boldsymbol{j}} u = 0$ if $\boldsymbol{j} \neq i^m$. Also there exists $\widetilde{u} \in L(i^m)$ such that $x_r^{m-1}\widetilde{u} \neq 0$ for all r. (We note that it is from these properties we may deduce Proposition 2.4.)

The induced module $\operatorname{Ind} L(i^m) \boxtimes L(j) \boxtimes L(i^n)$ has a weight basis

$$B = \left\{ \psi_{\widehat{w}} \otimes (u_r \otimes v \otimes y_s) \;\middle|\; 1 \leqslant r \leqslant m!,\ 1 \leqslant s \leqslant n!,\ w \in S_{m+1+n}/S_m \times S_1 \times S_n \right\} \tag{6.49}$$

as in Remark 2.1.

Proposition 6.12. *Let* $K = \operatorname{span}\{\psi_{\widehat{w}} \otimes (u_r \otimes v \otimes y_s) \in B \mid \ell(w) \neq 0\}$. *Suppose* $c = m + n \leqslant a$. *Then*

1. K *is a proper submodule of* $\operatorname{Ind} L(i^m) \boxtimes L(j) \boxtimes L(i^n)$.
2. *The quotient module* $\operatorname{Ind} L(i^m) \boxtimes L(j) \boxtimes L(i^n)/K$ *is irreducible with character* $[m]_i^! [n]_i^! i^m j i^n$.

Proof. It suffices to show

$$h\psi_{\widehat{w}} \otimes (u_r \otimes v \otimes y_s) \in K \tag{6.50}$$

where $\ell(w) > 0$ as h ranges over the generators $1_{\boldsymbol{j}}$, x_r, ψ_r of $R(\nu)$.

Considering the relations in Section 1.1.3, $h\psi_{\widehat{w}} \otimes (u_r \otimes v \otimes y_s)$ is 0 or a sum of terms of the form $\psi_{\widehat{w'}} \otimes (u' \otimes v \otimes y')$ with $\ell(w') \geqslant \ell(w) - 2$, so in other words, we reduce to the case $\ell(w) = 1$ or $\ell(w) = 2$ (or else the terms are obviously in K). In fact, it is only in considering relation (1.29) we examine $\ell(w) = 2$, and otherwise we examine $\ell(w) = 1$.

To make this reduction valid, we first examine the case $h = x_t$. Let $\boldsymbol{i} = \boldsymbol{i}(m,n)$. We first observe that for $w \in S_{m+1+n}/S_m \times S_1 \times S_n$, $w(m+1) = r+1$ if and only if $w(\boldsymbol{i}) = \boldsymbol{i}(r, c-r)$. In this case, we can factor $w = \tau\gamma$ with $\ell(w) = \ell(\tau) + \ell(\gamma)$ where γ is minimal such that $\gamma(\boldsymbol{i}) = \boldsymbol{i}(r, c-r)$. In particular $\gamma = s_{r+1} \dots s_{m-1} s_m$ or $\gamma = s_r \dots s_{m+2} s_{m+1}$, which has length $|m - r|$. By relation (1.30)

$$x_t \psi_{\widehat{w}} 1_{\boldsymbol{i}} = 1_{\boldsymbol{i}(r,c-r)} x_t \psi_{\widehat{w}} = 1_{\boldsymbol{i}(r,c-r)} \psi_{\widehat{w}} x_{w^{-1}(t)} + 1_{\boldsymbol{i}(r,c-r)} \sum \psi_{i_1} \dots \psi_{i_k}$$

where the sum is over some subset of (not necessarily reduced) subwords $s_{i_1} \dots s_{i_k}$ of $\widehat{w}$, all satisfying that if $z = s_{i_1} \dots s_{i_k}$ then $z(m+1) = r+1$. In particular $\ell(z) \geqslant |m-r|$. This shows (6.50) holds for $h = x_t$ when $\ell(w) > 0$.

For $h = 1_{\boldsymbol{j}}$, either $h\psi_{\widehat{w}} 1_{\boldsymbol{i}(m,n)} = 0$ or $h\psi_{\widehat{w}} 1_{\boldsymbol{i}(m,n)} = \psi_{\widehat{w}} 1_{\boldsymbol{i}(m,n)}$, so clearly (6.50) holds.

For $h = \psi_b$, when employing relation (1.29), we see some terms in $h\psi_{\widehat{w}} 1_{\boldsymbol{i}}$ may involve terms of the form $f(x_1, \dots, x_{c+1})\psi_{\widehat{w'}}$ with $\ell(w') = \ell(w) - 2$. However from the case completed above regarding relation (1.30), these terms still have length > 0 as long as $\ell(w') > 0$. In other words, we need only to consider the case $\ell(w) = 2$, for which either $w = s_{m\pm1}s_m$ or $w = s_{m+1\pm1}s_{m+1}$. However, the only cases that are potentially "length-decreasing" by 2 are for $w = s_{m+1}s_m$ and $h = \psi_m$, or $w = s_m s_{m+1}$ and $h = \psi_{m+1}$, for which we compute

$$(\psi_m\psi_{m+1}\psi_m - \psi_{m+1}\psi_m\psi_{m+1})1_{\boldsymbol{i}} = \sum_{k=0}^{a+1} x_m^k x_{m+2}^{a+1-k} 1_{\boldsymbol{i}}. \tag{6.51}$$

By (6.48)

$$x_m^k x_{m+2}^{a+1-k} \otimes (u \otimes v \otimes y) = 1_{\boldsymbol{i}} \otimes \left(x_m^k u\right) \otimes v \otimes \left(x_1^{a+1-k} y\right) = 0 \tag{6.52}$$

since either $k \geqslant m$ or $a+1-k > a+1-m \geqslant n$ as we assumed $m+n \leqslant a$. This yields

$$\psi_m\psi_{m+1}\psi_m \otimes (u \otimes v \otimes y) = \psi_{m+1}\psi_m\psi_{m+1} \otimes (u \otimes v \otimes y).$$

In fact, we also have $\psi_m\psi_{m-1}\psi_m \otimes (u \otimes v \otimes y) = \psi_{m-1}\psi_m\psi_{m-1} \otimes (u \otimes v \otimes y)$, as for instance $\boldsymbol{i}_{m-1} \neq \boldsymbol{i}_{m+1}$, and similarly $\psi_{m+1}\psi_{m+2}\psi_{m+1} \otimes (u \otimes v \otimes y) = \psi_{m+2}\psi_{m+1}\psi_{m+2} \otimes (u \otimes v \otimes y)$. Thus in all cases, this braid relation honestly holds. This then reduces us to the case $\ell(w) = 1$ as such relations decrease length by at most 1. For example,

$$\begin{aligned}\psi_m\psi_{m-1}\psi_m \otimes (u \otimes v \otimes y) &= \psi_{m-1}\psi_m\psi_{m-1} \otimes (u \otimes v \otimes y)\\ &= \psi_{m-1}\psi_m \otimes \left(u' \otimes v \otimes y\right).\end{aligned} \tag{6.53}$$

When $\ell(w) = 1$ either $w = s_m$ or $w = s_{m+1}$. For $h = \psi_b$ the only remaining relation that is length-decreasing is (1.28) (which decreases length by at most one, when $b = m$ or $m+1$), for which we compute

$$\begin{aligned}\psi_m\psi_m \otimes (u \otimes v \otimes y) &= \left(x_m^a + x_{m+1}^{-\langle j,i\rangle}\right)1_{\boldsymbol{i}} \otimes (u \otimes v \otimes y)\\ &= 1_{\boldsymbol{i}} \otimes \left(x_m^a u\right) \otimes v \otimes y + 1_{\boldsymbol{i}} \otimes u \otimes \left(x_1^{-\langle j,i\rangle} v\right) \otimes y\\ &= 0 \in K\end{aligned} \tag{6.54}$$

by (6.48) since $a \geqslant m$, and $-\langle j,i\rangle \geqslant 1$. Similarly,

$$\begin{aligned}\psi_{m+1}\psi_{m+1} \otimes (u \otimes v \otimes y) &= 1_{\boldsymbol{i}} \otimes u \otimes \left(x_1^{-\langle j,i\rangle} v\right) \otimes y + 1_{\boldsymbol{i}} \otimes u \otimes v \otimes \left(x_1^a y\right)\\ &= 0 \in K\end{aligned} \tag{6.55}$$

as $a \geqslant n$.

In conclusion, K is indeed a submodule and in fact generated by

$$\psi_{m+1} \otimes (u_r \otimes v \otimes y_s) \quad \text{and} \quad \psi_m \otimes (u_r \otimes v \otimes y_s). \tag{6.56}$$

For part 2 note $w(\boldsymbol{i}) = \boldsymbol{i}(c-r,r)$ for some r, but $r \neq n$ when $\ell(w) > 0$ for minimal length $w \in S_{m+1+n}/S_m \times S_1 \times S_n$. In other words, $\psi_{\widehat{w}} \otimes (u_r \otimes v \otimes y_s)$ is a weight vector and $1_{\boldsymbol{i}}\psi_{\widehat{w}} \otimes (u_r \otimes v \otimes y_s) = 0$ when $\ell(w) > 0$. That is, for all $z \in Q = \operatorname{Ind} L(i^m) \boxtimes L(j) \boxtimes L(i^n)/K$, $1_{\boldsymbol{i}} z = z$, but $1_{\boldsymbol{i}(c-r,r)} z = 0$ when $r \neq n$. Hence all constituents of $\operatorname{ch}(Q)$ have the form $i^m j i^n$.

By Frobenius reciprocity, and the irreducibility of $L(i^m)$, we have an injection

$$L\big(i^m\big) \boxtimes L(j) \boxtimes L\big(i^n\big) \hookrightarrow \operatorname{Res}_{mi,j,ni} Q \tag{6.57}$$

which is also a surjection by the above arguments. Hence

$$\operatorname{ch}(Q) = [m]_i^! [n]_i^! i^m j i^n. \tag{6.58}$$

Note that, up to grading shift, Q is none other than $\mathcal{L}(i^m j i^n)$ and we have shown this is the unique simple quotient of $\operatorname{Ind} L(i^m) \boxtimes L(j) \boxtimes L(i^n)$. The uniqueness statements of Theorem 6.10 follow by Frobenius reciprocity. $\square$

Next we will give the generators and relations proof that

$$\widetilde{f}_i \mathcal{L}(n) \cong \widetilde{f}_i^{\vee} \mathcal{L}(n) \cong \operatorname{Ind} \mathcal{L}(n) \boxtimes L(i). \tag{6.59}$$

Just as in the proof of Theorem 6.10,

$$\begin{aligned} &\operatorname{ch}\big(\operatorname{Ind} \mathcal{L}(n) \boxtimes L(i)\big) \\ &\quad = [a-n]_i^! [n+1]_i^! i^{a-n} j i^{n+1} + q^{-(\alpha_i,\alpha_j)} [a-n+1]_i^! [n]_i^! i^{a+n+1} j i^n, \end{aligned} \tag{6.60}$$

and since $L(i^m)$ is irreducible with dimension $m!$, either $\operatorname{ch}(\widetilde{f}_i \mathcal{L}(n)) = [a-n]_i^! [n+1]_i^! i^{a-n} j i^{n+1}$ or $\operatorname{ch}(\widetilde{f}_i \mathcal{L}(n)) = \operatorname{ch}(\operatorname{Ind} \mathcal{L}(n) \boxtimes L(i))$.

In the latter case, $\operatorname{Ind} \mathcal{L}(n) \boxtimes L(i)$ is isomorphic to $\widetilde{f}_i \mathcal{L}(n)$, so by the Jump Lemma 6.5 it is irreducible and isomorphic to $\widetilde{f}_i^{\vee} \mathcal{L}(n)$. In the former case, we clearly have

$$0 \to K \to \operatorname{Ind} L\big(i^{a-n}\big) \boxtimes L(j) \boxtimes L\big(i^{n+1}\big) \to \widetilde{f}_i \mathcal{L}(n) \tag{6.61}$$

by Frobenius reciprocity.

The $R((a+1)i+j)$-module $\operatorname{Ind} L(i^{a-n}) \boxtimes L(j) \boxtimes L(i^{n+1})$ has a weight basis given by

$$\big\{\psi_{\widehat{w}} \otimes (u_r \otimes v \otimes y_s) \bigm| w \in S_{a+2}/S_{a-n} \times S_1 \times S_{n+1},\ 1 \leqslant r \leqslant (a-n)!,\ 1 \leqslant s \leqslant (n+1)!\big\}. \tag{6.62}$$

Let $\boldsymbol{i} = \boldsymbol{i}(a-n, n+1)$. Note, for all minimal left coset representatives $w \in S_{a+2}/S_{a-n} \times S_1 \times S_{n+1}$ that $w(\boldsymbol{i}) \neq \boldsymbol{i}$ unless $w = \mathrm{id}$, i.e. unless $\ell(w) = 0$. (In fact $w(\boldsymbol{i}) = \boldsymbol{i}(a-r+1, r)$ for some r.)

Since $1_{\boldsymbol{i}(a-r+1,r)}\widetilde{f}_i\mathcal{L}(n)=0$ if $r\neq n+1$ by assumption, we must have

$$K=\operatorname{span}\bigl\{\psi_{\widehat{w}}\otimes(u_r\otimes v\otimes y_s)\bigm|\ell(w)>0\bigr\}. \tag{6.63}$$

We will show that K is not a proper submodule.

Pick $u\in L(i^{a-n})$, $y\in L(i^{n+1})$ so that $x_{a-n}^{a-n-1}u=u'\neq 0$, $x_1^n y=y'\neq 0$ so that

$$x_{a-n}^{a-n-1}\cdot x_{a-n+2}^{n}\bigl(1_{\boldsymbol{i}}\otimes(u\otimes v\otimes y)\bigr)=1_{\boldsymbol{i}}\otimes\bigl(u'\otimes v\otimes y'\bigr)\neq 0, \tag{6.64}$$

but

$$x_{a-n}^{a-1-k}u=0\quad\text{if }k<n \tag{6.65}$$

and

$$x_1^k y=0\quad\text{if }k>n. \tag{6.66}$$

Also recall u' generates $L(i^{a-n})$ and y' generates $L(i^{n+1})$ so $1_{\boldsymbol{i}}\otimes(u'\otimes v\otimes y')$ generates the module $\operatorname{Ind}L(i^{a-n})\boxtimes L(j)\boxtimes L(i^{n+1})$. By assumption, $K\ni\psi_{a-n+1}\otimes(u\otimes v\otimes y)$ and $K\ni\psi_{a-n}\otimes(u\otimes v\otimes y)$.

If K is an $R((a+1)i+j)$-submodule, K also contains

$$\begin{aligned}&(\psi_{a-n+1}\psi_{a-n}\psi_{a-n+1}-\psi_{a-n}\psi_{a-n+1}\psi_{a-n})\otimes(u\otimes v\otimes y)\\&\quad\overset{(1.29)}{=}\left(\sum_{k=0}^{a-1}x_{a-n}^{a-1-k}x_{a-n+2}^{k}\right)\otimes(u\otimes v\otimes y)\overset{(6.63),(6.64),(6.66)}{=}0+1_{\boldsymbol{i}}\otimes\bigl(u'\otimes v\otimes y'\bigr)\neq 0.\end{aligned}$$

Therefore $K\ni 1_{\boldsymbol{i}}\otimes(u'\otimes v\otimes y')$, hence K contains all of $\operatorname{Ind}L(i^{a-n})\boxtimes L(j)\boxtimes L(i^{n+1})$ contradicting that K is a proper submodule. We must have $\widetilde{f}_i\mathcal{L}(n)\cong\operatorname{Ind}\mathcal{L}(n)\boxtimes L(i)$. Now (6.39) in Theorem 6.11 follows for general m from the $m=1$ case as before.

Note that the Structure Theorems do not depend on the characteristic of $\Bbbk$. Just as the dimensions of simple $R(mi)$-modules are independent of $\operatorname{char}\Bbbk$, so are the dimensions of simple $R(ci+j)$-modules. In fact, Kleshchev and Ram have conjectured [38] that the dimensions of all simple $R(\nu)$-modules are independent of $\operatorname{char}\Bbbk$ for finite Cartan datum.

6.4. Understanding φ_i^{Λ}

The main theorems in this section measure how the crystal data differs for M and $\widetilde{f}_j M$. In particular, Theorem 6.21 below is equivalent to

$$\varphi_i^{\Lambda}(\widetilde{f}_j M)-\varepsilon_i(\widetilde{f}_j M)=a+\bigl(\varphi_i^{\Lambda}(M)-\varepsilon_i(M)\bigr) \tag{6.67}$$

where $a=-\langle h_i,\alpha_j\rangle$.

First we introduce several lemmas that will be needed.

Lemma 6.13. *Suppose $c+d \leqslant a$.*

(i) $\operatorname{Ind}\mathcal{L}(i^c j i^d) \boxtimes L(i^m)$ *has irreducible cosocle equal to*

$$\widetilde{f}_i^{\,m}\mathcal{L}\big(i^c j i^d\big) = \widetilde{f}_i^{\,m+d}\mathcal{L}\big(i^c j\big) = \begin{cases} \operatorname{Ind}\mathcal{L}(a-c) \boxtimes L(i^{m-a+c+d}), & m \geqslant a-(c+d), \\ \mathcal{L}(i^c j i^{d+m}), & m < a-(c+d). \end{cases} \tag{6.68}$$

(ii) *Suppose there is a nonzero map*

$$\operatorname{Ind}\mathcal{L}(c_1) \boxtimes \mathcal{L}(c_2) \boxtimes \cdots \boxtimes \mathcal{L}(c_r) \boxtimes L\big(i^m\big) \to Q \tag{6.69}$$

where Q is irreducible. Then $\varepsilon_i(Q) = m + \sum_{t=1}^r c_t$ and $\varepsilon_i^\vee(Q) = m + \sum_{t=1}^r (a-c_t)$.

(iii) *Let B and Q be irreducible and suppose there is a nonzero map $\operatorname{Ind} B \boxtimes \mathcal{L}(c) \to Q$. Then $\varepsilon_i(Q) = \varepsilon_i(B) + c$.*

Proof. Part (i) follows from the Structure Theorems 6.10, 6.11 for irreducible $R((c+d+m)i+j)$-modules. For part (ii) recall $\operatorname{Ind}\mathcal{L}(c) \boxtimes L(i^m)$ is irreducible and is isomorphic to $\operatorname{Ind} L(i^m) \boxtimes \mathcal{L}(c)$ by part (i) of Theorem 6.11. Consider the chain of homogeneous surjections

$$\begin{array}{c} \operatorname{Ind}\mathcal{L}(i^{a-c_1} j) \boxtimes \mathcal{L}(c_2) \boxtimes \cdots \boxtimes \mathcal{L}(c_r) \boxtimes L(i^{c_1+m}) \\ \Big\downarrow \cong \\ \operatorname{Ind}\mathcal{L}(i^{a-c_1} j) \boxtimes L(i^{c_1}) \boxtimes \mathcal{L}(c_2) \boxtimes \cdots \boxtimes \mathcal{L}(c_r) \boxtimes L(i^m) \\ \big\downarrow \\ \operatorname{Ind}\mathcal{L}(c_1) \boxtimes \mathcal{L}(c_2) \boxtimes \cdots \boxtimes \mathcal{L}(c_r) \boxtimes L(i^m) \\ \big\downarrow \\ Q. \end{array} \tag{6.70}$$

Iterating this process we get a surjection

$$\operatorname{Ind}\mathcal{L}\big(i^{a-c_1} j\big) \boxtimes \mathcal{L}\big(i^{a-c_2} j\big) \boxtimes \cdots \boxtimes \mathcal{L}\big(i^{a-c_r} j\big) \boxtimes L\big(i^h\big) \twoheadrightarrow Q \tag{6.71}$$

where $h = m + \sum_{t=1}^r c_t$. This shows that $\varepsilon_i(Q) = m + \sum_{t=1}^r c_t$. The computation of $\varepsilon_i^\vee(Q)$ is similar.

For part (iii) let $b = \varepsilon_i(B)$. By the Shuffle Lemma $\varepsilon_i(Q) \leqslant b+c$. Further there exists an irreducible module C such that $\varepsilon_i(C) = 0$ and $\operatorname{Ind} C \boxtimes L(i^b) \twoheadrightarrow B$. By the exactness of induction, we have a surjection

$$\operatorname{Ind} C \boxtimes \mathcal{L}(c) \boxtimes L\big(i^b\big) \cong \operatorname{Ind} C \boxtimes L\big(i^b\big) \boxtimes \mathcal{L}(c) \twoheadrightarrow Q \tag{6.72}$$

and so by Frobenius reciprocity $\varepsilon_i(Q) \geqslant \varepsilon_i(\mathcal{L}(c)) + \varepsilon_i(L(i^b)) = c + b$. □

Lemma 6.14. *Let N be an irreducible $R(ci + dj)$-module with $\varepsilon_i(N) = 0$. Suppose $c + d > 0$.*

(i) *There exists irreducible $\overline{N}$ with $\varepsilon_i(\overline{N}) = 0$ and a surjection*

$$\operatorname{Ind} \overline{N} \boxtimes \mathcal{L}\big(i^b j\big) \twoheadrightarrow N \tag{6.73}$$

with $b \leqslant a$.

(ii) *There exists an $r \in \mathbb{N}$ and $b_t \leqslant a$ for $1 \leqslant t \leqslant r$ such that*

$$\operatorname{Ind} \mathcal{L}\big(i^{b_1} j\big) \boxtimes \mathcal{L}\big(i^{b_2} j\big) \boxtimes \cdots \boxtimes \mathcal{L}\big(i^{b_r} j\big) \twoheadrightarrow N. \tag{6.74}$$

Proof. First, we may assume $\widetilde{e}_j N \neq \mathbf{0}$ or else N would be the trivial module $\mathbb{1}$, i.e. $c = d = 0$. Let $b = \varepsilon_i(\widetilde{e}_j N)$ and let $\overline{N} = \widetilde{e}_i^{\,b} \widetilde{e}_j N$ so that $\varepsilon_i(\overline{N}) = 0$. There exists a surjection

$$\operatorname{Ind} \overline{N} \boxtimes L\big(i^b\big) \boxtimes L(j) \twoheadrightarrow N. \tag{6.75}$$

Recall $\varepsilon_i(N) = 0$ and by the Structure Theorems, $\operatorname{Ind} L(i^b) \boxtimes L(j)$ has at most one composition factor with $\varepsilon_i = 0$, namely $\mathcal{L}(i^b j)$ in the case $b \leqslant a$. In the case $b > a$ it has no such composition factors, contradicting $\varepsilon_i(N) = 0$. Hence $b \leqslant a$ and the above map must factor through

$$\operatorname{Ind} \overline{N} \boxtimes \mathcal{L}\big(i^b j\big) \twoheadrightarrow N. \tag{6.76}$$

For part (ii) we merely repeat the argument from part (i) using the exactness of induction. □

Lemma 6.15. *Suppose Q is irreducible and we have a surjection*

$$\operatorname{Ind} \mathcal{L}\big(i^{b_1} j\big) \boxtimes \mathcal{L}\big(i^{b_2} j\big) \boxtimes \cdots \boxtimes \mathcal{L}\big(i^{b_r} j\big) \boxtimes L\big(i^h\big) \twoheadrightarrow Q. \tag{6.77}$$

(i) *Then for $h \gg 0$ we have a surjection*

$$\operatorname{Ind} \mathcal{L}(a - b_1) \boxtimes \mathcal{L}(a - b_2) \boxtimes \cdots \boxtimes \mathcal{L}(a - b_r) \boxtimes L\big(i^g\big) \twoheadrightarrow Q \tag{6.78}$$

where $g = h - \sum_{t=1}^{r} (a - b_t)$.

(ii) *In the case $h < ar - \sum_{t=1}^{r} b_t$, we have*

$$\operatorname{Ind} \mathcal{L}\big(i^{b_1} j\big) \boxtimes \cdots \boxtimes \mathcal{L}\big(i^{b_{s-1}} j\big) \boxtimes \mathcal{L}\big(i^{b_s} j i^{g'}\big) \boxtimes \mathcal{L}(a - b_{s+1}) \boxtimes \cdots \boxtimes \mathcal{L}(a - b_r) \twoheadrightarrow Q \tag{6.79}$$

where $g' = h - \sum_{t=s+1}^{r} (a - b_t)$ and s is such that

$$\sum_{t=s+1}^{r} (a - b_t) \leqslant h < \sum_{t=s}^{r} (a - b_t). \tag{6.80}$$

Proof. Observe that $\varepsilon_i(Q) = h$. Similar to Lemma 6.13(i) when $d = 0$, $\operatorname{Ind}\mathcal{L}(i^{b_r} j) \boxtimes L(i^h)$ has a unique composition factor with $\varepsilon_i = h$, namely $\operatorname{Ind} L(i^{h-(a-b_r)}) \boxtimes \mathcal{L}(a - b_r)$ in the case $h \geqslant a - b_r$ and $\mathcal{L}(i^{b_r} j i^h)$ otherwise. In the latter case, we are done, and note we fall into case (ii) with $s = r$. In the former case, we get a surjection

$$\operatorname{Ind}\mathcal{L}\big(i^{b_1} j\big) \boxtimes \cdots \boxtimes \mathcal{L}\big(i^{b_{r-1}} j\big) \boxtimes L\big(i^{h-(a-b_r)}\big) \boxtimes \mathcal{L}(a - b_r) \twoheadrightarrow Q. \tag{6.81}$$

We apply the same reasoning to $\operatorname{Ind}\mathcal{L}(i^{b_{r-1}} j) \boxtimes L(i^{h-(a-b_r)})$ noting that by Lemma 6.13(iii), since $\varepsilon_i(\mathcal{L}(a - b_r)) = a - b_r = \varepsilon_i(Q) - (h - (a - b_r))$ we want to pick out the unique composition factor with $\varepsilon_i = h - (a - b_r)$. As above, this is $\operatorname{Ind} L(i^{h-\sum_{t=r-1}^{r} b_t}) \boxtimes \mathcal{L}(a - b_{r-1})$ for h large enough and $\mathcal{L}(i^{b_{r-1}} j i^{h-(a-b_r)})$ otherwise. Continuing in this vein the lemma follows. $\square$

Lemma 6.16. *Let M be an irreducible $R(\nu)$-module and suppose we have a nonzero map*

$$\operatorname{Ind} A \boxtimes B \boxtimes L\big(i^h\big) \xrightarrow{f} M \tag{6.82}$$

where $\varepsilon_i(A) = 0$ and B is irreducible. Then there exists a surjective map

$$\operatorname{Ind} A \boxtimes \widetilde{f}_i^{\,h} B \twoheadrightarrow M. \tag{6.83}$$

Proof. First note $\varepsilon_i(M) = \varepsilon_i(B) + h$ since by Frobenius reciprocity $\varepsilon_i(M) \geqslant \varepsilon_i(B) + h$, but by the Shuffle Lemma $\varepsilon_i(M) \leqslant \varepsilon_i(B) + h$ since $\varepsilon_i(A) = 0$. Consider $\operatorname{Ind} B \boxtimes L(i^h)$. This has unique irreducible quotient $\widetilde{f}_i^{\,h} B$ with $\varepsilon_i(\widetilde{f}_i^{\,h} B) = \varepsilon_i(B) + h$ and has all other composition factors U with $\varepsilon_i(U) < \varepsilon_i(B) + h = \varepsilon_i(M)$, by Section 2.5.1. Hence, for any such U there does not exist a nonzero map $\operatorname{Ind} A \boxtimes U \to M$. In particular, letting K be the maximal submodule such that

$$0 \to K \to \operatorname{Ind} B \boxtimes L\big(i^h\big) \to \widetilde{f}_i^{\,h} B \to 0 \tag{6.84}$$

is exact, the above map f must restrict to zero on the submodule $\operatorname{Ind} A \boxtimes K$ and hence f factors through $\operatorname{Ind} A \boxtimes \widetilde{f}_i^{\,h} B \twoheadrightarrow M$, which is nonzero and thus surjective. $\square$

Lemma 6.17. *Let A be an irreducible $R(\nu)$-module with $\operatorname{pr}_\Lambda A \neq \mathbf{0}$ and $k = \varphi_i^\Lambda(A)$.*

(i) *Let U be an irreducible $R(\mu)$-module and let $t \geqslant 1$. Then $\operatorname{pr}_\Lambda \operatorname{Ind} A \boxtimes L(i^{k+t}) \boxtimes U = \mathbf{0}$.*
(ii) *Let B be irreducible with $\varepsilon_i^\vee(B) > k$. Then $\operatorname{pr}_\Lambda \operatorname{Ind} A \boxtimes B = \mathbf{0}$. In particular, if Q is any irreducible quotient of $\operatorname{Ind} A \boxtimes B$, then $\operatorname{pr}_\Lambda Q = \mathbf{0}$.*

Proof. Recall for a module B, $\operatorname{pr}_\Lambda B = B/\mathcal{J}^\Lambda B$ and so $\operatorname{pr}_\Lambda B = \mathbf{0}$ if and only if $B = \mathcal{J}^\Lambda B$. Since A, $L(i^{k+t})$, and U are all irreducible, each is generated by any single nonzero element. Let us pick nonzero $w \in A$, $v \in L(i^{k+t})$, $u \in U$. Further $\operatorname{Ind} A \boxtimes L(i^{k+t})$ is cyclically generated as an $R(\nu + (k + t)i)$-module by $1_{\nu+(k+t)i} \otimes w \otimes v$ and likewise $\operatorname{Ind} A \boxtimes L(i^{k+t}) \boxtimes U$ is generated as an $R(\nu + (k + t)i + \mu)$-module by $1_{\nu+(k+t)i+\mu} \otimes w \otimes v \otimes u$.

Recall that $\operatorname{Ind} A \boxtimes L(i^{k+t})$ has a unique simple quotient $\widetilde{f}_i^{\,k+t} A$ and that $\operatorname{pr}_\Lambda \widetilde{f}_i^{\,k+t} A = \mathbf{0}$ because $\varphi_i^\Lambda(A) = k$. Since $\operatorname{pr}_\Lambda$ is right exact, $\operatorname{pr}_\Lambda \operatorname{Ind} A \boxtimes L(i^{k+t}) = \mathbf{0}$. Consequently,

$\mathcal{J}^{\Lambda}_{\nu+(k+t)i}\operatorname{Ind} A \boxtimes L(i^{k+t}) = \operatorname{Ind} A \boxtimes L(i^{k+t})$. In particular, there exists an $\eta \in \mathcal{J}^{\Lambda}_{\nu+(k+t)i}$ such that

$$\eta 1_{\nu+(k+t)i} \otimes w \otimes v = 1_{\nu+(k+t)i} \otimes w \otimes v. \tag{6.85}$$

But then

$$\eta 1_{\nu+(k+t)i+\mu} \otimes w \otimes v \otimes u = 1_{\nu+(k+t)i+\mu} \otimes w \otimes v \otimes u. \tag{6.86}$$

Note that we can consider η as an element of $\mathcal{J}^{\Lambda}_{\nu+(k+t)i+\mu}$ as well via the canonical inclusion $R(\nu+(k+t)i) \hookrightarrow R(\nu+(k+t)i+\mu)$. Hence

$$\mathcal{J}^{\Lambda}_{\nu+(k+t)i+\mu}\operatorname{Ind} A \boxtimes L\big(i^{k+t}\big) \boxtimes U = \operatorname{Ind} A \boxtimes L\big(i^{k+t}\big) \boxtimes U \tag{6.87}$$

and so $\operatorname{pr}_{\Lambda}\operatorname{Ind} A \boxtimes L(i^{k+t}) \boxtimes U = \mathbf{0}$.

For part (ii), let $b = \varepsilon_i^{\vee}(B)$ and $C = (\widetilde{e}_i^{\,\vee})^b B$ so we have $\operatorname{Ind} L(i^b) \boxtimes C \twoheadrightarrow B$. Thus by the exactness of induction we also have a surjection $\operatorname{Ind} A \boxtimes L(i^b) \boxtimes C \twoheadrightarrow \operatorname{Ind} A \boxtimes B$. By part (i) and the right exactness of $\operatorname{pr}_{\Lambda}$, $\operatorname{pr}_{\Lambda}\operatorname{Ind} A \boxtimes B = \mathbf{0}$. Likewise $\operatorname{pr}_{\Lambda} Q = \mathbf{0}$ for any quotient of $\operatorname{Ind} A \boxtimes B$. $\square$

Lemma 6.18. *Let A be an irreducible $R(\nu)$-module with $\operatorname{pr}_{\Lambda} A \neq \mathbf{0}$ and $k = \varphi_i^{\Lambda}(A)$. Further suppose $\varepsilon_i(A) = \varepsilon_j(A) = 0$ and that B is an irreducible $R(ci+dj)$-module with $\varepsilon_i^{\vee}(B) \leqslant k$. Let Q be irreducible such that $\operatorname{Ind} A \boxtimes B \twoheadrightarrow Q$ is nonzero. Then $\varepsilon_i^{\vee}(Q) \leqslant \lambda_i$. Further, if $\varepsilon_j^{\vee}(B) \leqslant \varphi_j^{\Lambda}(A)$ (or if $\lambda_j \gg 0$) then $\operatorname{pr}_{\Lambda} Q \neq \mathbf{0}$.*

Proof. Let $b = \varepsilon_i^{\vee}(B)$ and $C = (\widetilde{e}_i^{\,\vee})^b B$ so that $\varepsilon_i^{\vee}(C) = 0$. We thus have surjections

$$\operatorname{Ind} A \boxtimes L\big(i^b\big) \boxtimes C \twoheadrightarrow \operatorname{Ind} A \boxtimes B \twoheadrightarrow Q. \tag{6.88}$$

Observe by Frobenius reciprocity

$$(1_{\nu} \otimes 1_{bi} \otimes 1_{(c-b)i+dj})Q \neq \mathbf{0}. \tag{6.89}$$

Let U be any composition factor of $\operatorname{Ind} A \boxtimes L(i^b)$ other than $\widetilde{f}_i^{\,b} A$, so that $\varepsilon_i(U) < b$. By the Shuffle Lemma $1_{\nu} \otimes 1_{bi} \otimes 1_{(c-b)i+dj}(\operatorname{Ind} U \boxtimes C) = \mathbf{0}$, so there cannot be a nonzero homomorphism $\operatorname{Ind} U \boxtimes C \twoheadrightarrow Q$. (More precisely, for every constituent $\boldsymbol{i} = i_1 \dots i_{|\nu|+b}$ of $\operatorname{ch}(U)$ there exists a y, $|\nu| < y \leqslant |\nu| + b$ with $i_y \neq i$ and $i_y \neq j$. Hence by the Shuffle Lemma, for every constituent $\boldsymbol{i}' = i'_1 \dots i'_{|\nu|+c+d}$ of $\operatorname{ch}(\operatorname{Ind} U \boxtimes C)$ there exists a z, $|\nu| < z \leqslant |\nu| + c + d$ with $i'_z \neq i$ and $i'_z \neq j$.)

Thus we must have a nonzero map

$$\operatorname{Ind} \widetilde{f}_i^{\,b} A \boxtimes C \twoheadrightarrow Q. \tag{6.90}$$

By the Shuffle Lemma, $\varepsilon_i^{\vee}(Q) \leqslant \varepsilon_i^{\vee}(\widetilde{f}_i^{\,b} A) + \varepsilon_i^{\vee}(C) \leqslant \lambda_i$ since $b \leqslant k = \varphi_i^{\Lambda}(A)$ and $\varepsilon_i^{\vee}(C) = 0$. Note $\varepsilon_{\ell}^{\vee}(Q) \leqslant \varepsilon_{\ell}^{\vee}(A) + \varepsilon_{\ell}^{\vee}(B)$, so for $\ell \neq i$, $\ell \neq j$ clearly $\varepsilon_{\ell}^{\vee}(Q) \leqslant \lambda_{\ell}$ and hence $\operatorname{pr}_{\Lambda} Q \neq \mathbf{0}$ as long as $\varepsilon_j^{\vee}(B) \leqslant \varphi_j^{\Lambda}(A)$, which will for instance be assured if $\lambda_j \gg 0$. $\square$

In the following theorem and its proof all modules have support $\nu = ci + dj$ for some $c, d \in \mathbb{N}$.

Theorem 6.19. *Let M be an irreducible $R(ci+dj)$-module and let $\Lambda \in P^+$ be such that $\mathrm{pr}_\Lambda M \neq \mathbf{0}$ and $\mathrm{pr}_\Lambda \widetilde{f}_j M \neq \mathbf{0}$. Let $m = \varepsilon_i(M)$, $k = \varphi_i^\Lambda(M)$. Then there exists an n with $0 \leqslant n \leqslant a$ such that $\varepsilon_i(\widetilde{f}_j M) = m - (a-n)$ and $\varphi_i^\Lambda(\widetilde{f}_j M) = k + n$.*

Proof. Let $N = \widetilde{e}_i^{\,m} M$ so that $\varepsilon_i(N) = 0$ and we have a surjection

$$\operatorname{Ind} N \boxtimes L\big(i^m\big) \twoheadrightarrow M. \tag{6.91}$$

Thus, we also have

$$\operatorname{Ind} N \boxtimes L\big(i^m\big) \boxtimes L(j) \twoheadrightarrow \widetilde{f}_j M. \tag{6.92}$$

By the Structure Theorems 6.10, 6.11 for simple $R(mi+j)$-modules, for each $m - a \leqslant \gamma \leqslant m$ there exists a composition factor U_γ of $\operatorname{Ind} L(i^m) \boxtimes L(j)$ with $\varepsilon_i(U_\gamma) = \gamma$. In particular, there is a unique γ such that the above map induces

$$\operatorname{Ind} N \boxtimes U_\gamma \twoheadrightarrow \widetilde{f}_j M \tag{6.93}$$

as we must have $\varepsilon_i(U_\gamma) = \varepsilon_i(\widetilde{f}_j M)$, since $\varepsilon_i(N) = 0$. Choose n so that $\gamma = m - (a-n) = \varepsilon_i(\widetilde{f}_j M)$. Note that by the Structure Theorems

$$U_\gamma \cong \begin{cases} \operatorname{Ind} \mathcal{L}(n) \boxtimes L(i^{m-a}), & m \geqslant a, \\ \mathcal{L}(i^{a-n} j i^{m-(a-n)}), & m < a, \end{cases} \tag{6.94}$$

and furthermore

$$\widetilde{f}_i^{\,a} U_\gamma \cong \operatorname{Ind} \mathcal{L}(n) \boxtimes L\big(i^m\big) \tag{6.95}$$

in both cases.

By Lemma 6.14 there exist $0 \leqslant b_t \leqslant a$ such that

$$\operatorname{Ind} \mathcal{L}\big(i^{b_1} j\big) \boxtimes \mathcal{L}\big(i^{b_2} j\big) \boxtimes \cdots \boxtimes \mathcal{L}\big(i^{b_r} j\big) \twoheadrightarrow N \tag{6.96}$$

and hence we obtain the following surjections

$$\operatorname{Ind} \mathcal{L}\big(i^{b_1} j\big) \boxtimes \mathcal{L}\big(i^{b_2} j\big) \boxtimes \cdots \boxtimes \mathcal{L}\big(i^{b_r} j\big) \boxtimes L\big(i^m\big) \twoheadrightarrow M, \tag{6.97}$$

$$\operatorname{Ind} \mathcal{L}\big(i^{b_1} j\big) \boxtimes \mathcal{L}\big(i^{b_2} j\big) \boxtimes \cdots \boxtimes \mathcal{L}\big(i^{b_r} j\big) \boxtimes L\big(i^{m+h}\big) \twoheadrightarrow \widetilde{f}_i^{\,h} M, \tag{6.98}$$

$$\operatorname{Ind} \mathcal{L}\big(i^{b_1} j\big) \boxtimes \mathcal{L}\big(i^{b_2} j\big) \boxtimes \cdots \boxtimes \mathcal{L}\big(i^{b_r} j\big) \boxtimes U_{m-a+n} \twoheadrightarrow \widetilde{f}_j M, \tag{6.99}$$

$$\operatorname{Ind} \mathcal{L}\big(i^{b_1} j\big) \boxtimes \mathcal{L}\big(i^{b_2} j\big) \boxtimes \cdots \boxtimes \mathcal{L}\big(i^{b_r} j\big) \boxtimes U_{m-a+n} \boxtimes L\big(i^h\big) \twoheadrightarrow \widetilde{f}_i^{\,h} \widetilde{f}_j M. \tag{6.100}$$

We first apply Lemma 6.15 to (6.98) to obtain, for $h \gg 0$ (in fact $h \geqslant \sum_{t=1}^r (a - b_t) - m$)

$$\operatorname{Ind} \mathcal{L}(a-b_1) \boxtimes \mathcal{L}(a-b_2) \boxtimes \cdots \boxtimes \mathcal{L}(a-b_r) \boxtimes L\big(i^g\big) \twoheadrightarrow \widetilde{f}_i^{\,h} M \tag{6.101}$$

where $g = m + h - \sum_{t=1}^{r}(a - b_t)$. Hence, by Lemma 6.13(ii)

$$\varepsilon_i^\vee\big(\widetilde{f_i}^h M\big) = g + \sum_{t=1}^{r} b_t = h + m - ar + 2\sum_{t=1}^{r} b_t. \tag{6.102}$$

Further, it is clear that $\varepsilon_i^\vee(\widetilde{f_i}^{h+1}) = 1 + \varepsilon_i^\vee(\widetilde{f_i}^h(M))$.

Applying Lemma 6.15 to (6.100) we obtain for $h \gg 0$

$$\operatorname{Ind} \mathcal{L}(a - b_1) \boxtimes \cdots \boxtimes \mathcal{L}(a - b_r) \boxtimes \mathcal{L}(n) \boxtimes L\big(i^m\big) \boxtimes L\big(i^{g'}\big) \twoheadrightarrow \widetilde{f_i}^h \widetilde{f_j} M \tag{6.103}$$

where $g' = h - a - \sum_{t=1}^{r}(a - b_t)$. Note that we have used (6.95) above, and in the case $m < a$ we have also employed Lemma 6.16. As above, by Lemma 6.13(ii)

$$\varepsilon_i^\vee\big(\widetilde{f_i}^h \widetilde{f_j} M\big) = g' + m + a - n + \sum_{t=1}^{r} b_t \tag{6.104}$$

$$= h + m - n - ar + 2\sum_{t=1}^{r} b_t \tag{6.105}$$

$$= \varepsilon_i^\vee\big(\widetilde{f_i}^h M\big) - n. \tag{6.106}$$

Further, it is clear that $\varepsilon_i^\vee(\widetilde{f_i}^{h+1} \widetilde{f_j} M) = 1 + \varepsilon_i^\vee(\widetilde{f_i}^h \widetilde{f_j} M)$.

For $h \gg 0$ we have shown that $\varepsilon_i^\vee(\widetilde{f_i}^h \widetilde{f_j} M) = \varepsilon_i^\vee(\widetilde{f_i}^h M) - n$. Now fix such an h and let $\omega_i = h + (m - ar + 2\sum_{t=1}^{r} b_t)$, which we may assume is positive. Let $\omega_\ell = \lambda_\ell$ for $\ell \neq i$ and set $\Omega = \sum_{i \in I} \omega_i \Lambda_i \in P^+$. Given these choices, we have shown $\varepsilon_i^\vee(\widetilde{f_i}^h M) = \omega_i$, but $\varepsilon_i^\vee(\widetilde{f_i}^{h+1} M) = \omega_i + 1$. Hence $\varphi_i^\Omega(M) = h$. Likewise $\varepsilon_i^\vee(\widetilde{f_i}^h \widetilde{f_j} M) = \omega_i - n$, so that $\varepsilon_i^\vee(\widetilde{f_i}^{h+n} \widetilde{f_j} M) = \omega_i$, but $\varepsilon_i^\vee(\widetilde{f_i}^{h+n+1} \widetilde{f_j} M) = \omega_i + 1$ yielding $\varphi_i^\Omega(\widetilde{f_j} M) = h + n$. Observe then that

$$\varphi_i^\Omega(\widetilde{f_j} M) - \varphi_i^\Omega(M) = n. \tag{6.107}$$

By our hypotheses and the choice of Ω, we know pr_Λ and pr_Ω are nonzero for both modules. Hence by Remark 6.8,

$$\varphi_i^\Lambda(\widetilde{f_j} M) - \varphi_i^\Lambda(M) = \varphi_i^\Omega(\widetilde{f_j} M) - \varphi_i^\Omega(M) = n. \qquad \square$$

We have just shown in Theorem 6.19 that Theorem 6.21 holds for all $R(ci + dj)$-modules. Next we show that to deduce the theorem for $R(\nu)$-modules for arbitrary ν it suffices to know the result for $\nu = ci + dj$.

Proposition 6.20. *Let $\Lambda \in P^+$ and let M be an irreducible $R(\nu)$-module such that $\mathrm{pr}_\Lambda M \neq \mathbf{0}$ and $\mathrm{pr}_\Lambda \widetilde{f_j} M \neq \mathbf{0}$. Suppose $\varepsilon_i(M) = m$ and $\varepsilon_i(\widetilde{f_j} M) = m - (a - n)$ for some $0 \leqslant n \leqslant a$. Then* *there exist c, d and an irreducible $R(ci + dj)$-module B such that $\varepsilon_i(B) = m$, $\varepsilon_i(\widetilde{f}_j B) = m - (a - n)$ and there exists $\Omega \in P^+$ with $\mathrm{pr}_\Omega(B) \neq \mathbf{0}$, $\mathrm{pr}_\Omega(\widetilde{f}_j B) \neq \mathbf{0}$, $\mathrm{pr}_\Omega(M) \neq \mathbf{0}$, $\mathrm{pr}_\Omega(\widetilde{f}_j M) \neq \mathbf{0}$, and furthermore*

$$\varphi_i^\Omega(\widetilde{f}_j M) - \varphi_i^\Omega(M) = \varphi_i^\Omega(\widetilde{f}_j B) - \varphi_i^\Omega(B). \tag{6.108}$$

Note that by Remark 6.8 $\varphi_i^\Lambda(\widetilde{f}_j M) - \varphi_i^\Lambda(M) = \varphi_i^\Omega(\widetilde{f}_j M) - \varphi_i^\Omega(M)$, so once we prove this proposition, it together with Theorem 6.19 proves Theorem 6.21.

Proof of Proposition 6.20. Let $N = \widetilde{e}_i^{\,m} M$, so that $\varepsilon_i(N) = 0$. Then there exist irreducible modules A and $\overline{B}$ with a surjection $\mathrm{Ind}\, A \boxtimes \overline{B} \twoheadrightarrow N$ such that $\varepsilon_i(A) = \varepsilon_j(A) = 0$ and $\overline{B}$ is an $R(\bar{c}i + dj)$-module for some $\bar{c}$, d. (For instance, one may construct A by setting

$$A_1 = N, \qquad A_{2r} = \widetilde{e}_j^{\,\varepsilon_j(A_{2r-1})} A_{2r-1}, \qquad A_{2r+1} = \widetilde{e}_i^{\,\varepsilon_i(A_{2r})} A_{2r} \tag{6.109}$$

which eventually stabilizes. So we may set $A = A_r$ for $r \gg 0$.)

Observe, as $\varepsilon_i(A) = \varepsilon_j(A) = 0$, we must have $\varepsilon_i(\overline{B}) = \varepsilon_i(N) = 0$ and $\varepsilon_j(\overline{B}) = \varepsilon_j(N)$. Hence we also have a surjection

$$\mathrm{Ind}\, A \boxtimes \overline{B} \boxtimes L(i^m) \twoheadrightarrow M \tag{6.110}$$

which by Lemma 6.16 produces a map

$$\mathrm{Ind}\, A \boxtimes B \twoheadrightarrow M \tag{6.111}$$

where $B = \widetilde{f}_i^{\,m} \overline{B}$. Observe $\varepsilon_i(B) = \varepsilon_i(M) = m$. We have a surjection

$$\mathrm{Ind}\, A \boxtimes B \boxtimes L(j) \twoheadrightarrow \widetilde{f}_j M \tag{6.112}$$

and since $\varepsilon_j(B) = \varepsilon_j(M)$, Lemma 6.16 again produces a map

$$\mathrm{Ind}\, A \boxtimes \widetilde{f}_j B \twoheadrightarrow \widetilde{f}_j M. \tag{6.113}$$

Again observe $\varepsilon_i(\widetilde{f}_j B) = \varepsilon_i(\widetilde{f}_j M) = m - (a - n)$. From (6.111) and (6.113) we also have nonzero maps

$$\mathrm{Ind}\, A \boxtimes B \boxtimes L(i^h) \twoheadrightarrow \widetilde{f}_i^{\,h} M, \qquad \mathrm{Ind}\, A \boxtimes \widetilde{f}_j B \boxtimes L(i^{h'}) \twoheadrightarrow \widetilde{f}_i^{\,h'} \widetilde{f}_j M \tag{6.114}$$

so applying Lemma 6.16, there exist surjections

$$\mathrm{Ind}\, A \boxtimes \widetilde{f}_i^{\,h} B \twoheadrightarrow \widetilde{f}_i^{\,h} M, \qquad \mathrm{Ind}\, A \boxtimes \widetilde{f}_i^{\,h'} \widetilde{f}_j B \twoheadrightarrow \widetilde{f}_i^{\,h'} \widetilde{f}_j M. \tag{6.115}$$

Let $\Omega = \sum_{i \in I} \omega_i \Lambda_i \in P^+$ be such that $\omega_\ell = \max\{\lambda_\ell, \varepsilon_\ell^\vee B\}$ for all $\ell \in I$. Recall B is an $R(ci + dj)$-module, where $c = \bar{c} + m$, so for $\ell \neq i, j$, $\varepsilon_\ell^\vee B = 0$. Take $h = \varphi_i^\Omega(M)$ and $h' = \varphi_i^\Omega(\widetilde{f}_j M)$ so that $\mathrm{pr}_\Omega(\widetilde{f}_i^{\,h} M) \neq \mathbf{0}$, $\mathrm{pr}_\Omega(\widetilde{f}_i^{\,h'} \widetilde{f}_j M) \neq \mathbf{0}$, but $\mathrm{pr}_\Omega(\widetilde{f}_i^{\,h+1} M) = \mathrm{pr}_\Omega(\widetilde{f}_i^{\,h'+1} \widetilde{f}_j M) = \mathbf{0}$.

From the contrapositive to Lemma 6.17(ii) applied to (6.115) we deduce

$$\varepsilon_i^\vee\big(\widetilde{f_i}^h B\big) \leqslant \varphi_i^\Omega(A), \qquad \varepsilon_i^\vee\big(\widetilde{f_i}^{h'}\widetilde{f_j}B\big) \leqslant \varphi_i^\Omega(A). \tag{6.116}$$

However, applying the contrapositive of Lemma 6.18

$$\varepsilon_i^\vee\big(\widetilde{f_i}^{h+1} B\big) > \varphi_i^\Omega(A), \qquad \varepsilon_i^\vee\big(\widetilde{f_i}^{h'+1}\widetilde{f_j}B\big) > \varphi_i^\Omega(A). \tag{6.117}$$

We thus conclude

$$\varepsilon_i^\vee\big(\widetilde{f_i}^h B\big) = \varphi_i^\Omega(A) = \varepsilon_i^\vee\big(\widetilde{f_i}^{h'}\widetilde{f_j}B\big) \tag{6.118}$$

and furthermore $\mathrm{jump}_i(\widetilde{f_i}^h B) = \mathrm{jump}_i(\widetilde{f_i}^{h'}\widetilde{f_j}B) = 0$.

Recall that $\varphi_i^\Omega(C) = 1 + \varphi_i^\Omega(\widetilde{f_i}C)$ for any irreducible module C. Hence, we compute

$$\begin{aligned}
\varphi_i^\Omega(\widetilde{f_j}B) - \varphi_i^\Omega(B) &= \big(h' + \varphi_i^\Omega\big(\widetilde{f_i}^{h'}\widetilde{f_j}B\big)\big) - \big(h + \varphi_i^\Omega\big(\widetilde{f_i}^{h}B\big)\big) \\
&= \big(h' - h\big) + \varphi_i^\Omega\big(\widetilde{f_i}^{h'}\widetilde{f_j}B\big) - \varphi_i^\Omega\big(\widetilde{f_i}^{h}B\big) \\
&\overset{\text{Prop. 6.6(ii)}}{=} \big(h' - h\big) + \big(\mathrm{jump}_i\big(\widetilde{f_i}^{h'}\widetilde{f_j}B\big) - \varepsilon_i^\vee\big(\widetilde{f_i}^{h'}\widetilde{f_j}B\big) + \omega_i\big) \\
&\quad - \big(\mathrm{jump}_i\big(\widetilde{f_i}^{h}B\big) - \varepsilon_i^\vee\big(\widetilde{f_i}^{h}B\big) + \omega_i\big) \\
&= \big(h' - h\big) + \big(0 - \varphi_i^\Omega(A) + \omega_i\big) - \big(0 - \varphi_i^\Omega(A) + \omega_i\big) \\
&= h' - h \\
&= \varphi_i^\Omega(\widetilde{f_j}M) - \varphi_i^\Omega(M). \qquad \square
\end{aligned}$$

Theorem 6.21. *Let M be an irreducible $R(\nu)$-module $\Lambda \in P^+$ such that $\mathrm{pr}_\Lambda M \neq \mathbf{0}$ and $\mathrm{pr}_\Lambda \widetilde{f_j}M \neq \mathbf{0}$. Let $m = \varepsilon_i(M)$, $k = \varphi_i^\Lambda(M)$. Then there exists an n with $0 \leqslant n \leqslant a$ such that $\varepsilon_i(\widetilde{f_j}M) = m - (a-n)$ and $\varphi_i^\Lambda(\widetilde{f_j}M) = k + n$.*

Proof. This follows from Theorem 6.19 which proves the theorem in the case $\nu = ci + dj$ and from Proposition 6.20 which reduces it to this case. $\square$

One important rephrasing of the theorem is

$$\begin{aligned}
\varphi_i^\Lambda(\widetilde{f_j}M) - \varepsilon_i(\widetilde{f_j}M) &= a + \big(\varphi_i^\Lambda(M) - \varepsilon_i(M)\big) \\
&= -\langle h_i, \alpha_j\rangle + \big(\varphi_i^\Lambda(M) - \varepsilon_i(M)\big).
\end{aligned} \tag{6.119}$$

Corollary 6.22. *Let $\Lambda = \sum_{i\in I}\lambda_i\Lambda_i \in P^+$ and let M be an irreducible $R(\nu)$-module such that $\mathrm{pr}_\Lambda M \neq \mathbf{0}$. Then*

$$\varphi_i^\Lambda(M) = \lambda_i + \varepsilon_i(M) + \mathrm{wt}_i(M).$$

Proof. The proof is by induction on the length $|\nu|$. For $|\nu| = 0$ we have $M = \mathbb{1}$ and $\mathrm{wt}(M) = 0$. For all $i \in I$ observe that $\varphi_i^\Lambda(\mathbb{1}) = \lambda_i$, $\varepsilon_i(\mathbb{1}) = 0$, and $\mathrm{wt}_i(M) = 0$, so that the claim clearly holds for $M = \mathbb{1}$. Fix ν with $|\nu| > 0$ and an irreducible $R(\nu)$-module M. Let $j \in I$ be such that $\varepsilon_j(M) \neq 0$, noting such j exists since $|\nu| > 0$.

Consider $N = \widetilde{e}_j M$. By induction we may assume the claim holds for N. Note $M = \widetilde{f}_j N$. By Theorem 6.21 and its rephrasing (6.119), for any $i \in I$

$$\begin{aligned}
\varphi_i^\Lambda(M) = \varphi_i^\Lambda(\widetilde{f}_j N) &= \varphi_i^\Lambda(N) + \varepsilon_i(\widetilde{f}_j N) - \varepsilon_i(N) + a_{ij} \\
&= \bigl(\lambda_i + \varepsilon_i(N) + \mathrm{wt}_i(N)\bigr) + \varepsilon_i(\widetilde{f}_j N) - \varepsilon_i(N) + a_{ij} \\
&= \lambda_i + \varepsilon_i(\widetilde{f}_j N) + \mathrm{wt}_i(N) - \langle h_i, \alpha_j \rangle \\
&= \lambda_i + \varepsilon_i(M) + \mathrm{wt}_i(M). \qquad \square
\end{aligned}$$

Note that we have finally proved Proposition 6.7(v). By Proposition 2.4, given an irreducible module M we can always take Λ large enough so that $\mathrm{pr}_\Lambda M \neq \mathbf{0}$, and then Proposition 6.6(ii) combined with the above corollary gives

$$\begin{aligned}
\mathrm{jump}_i(M) &= \varphi_i^\Lambda(M) + \varepsilon_i^\vee(M) + \lambda_i \\
&= \bigl(\lambda_i + \varepsilon_i(M) + \mathrm{wt}_i(M)\bigr) + \varepsilon_i^\vee(M) - \lambda_i \\
&= \varepsilon_i(M) + \varepsilon_i^\vee(M) + \mathrm{wt}_i(M).
\end{aligned} \tag{6.120}$$

As mentioned in the discussion below Proposition 6.7, the σ-symmetry of this characterization of $\mathrm{jump}_i(M)$ now implies the remaining parts (iii), (iv) of that proposition. In the next section, we will use all characterizations of $\mathrm{jump}_i(M)$ from Propositions 6.6 and 6.7.

7. Identification of crystals – "reaping the harvest"

Now that we have built up the machinery of Section 6, we can prove the module theoretic crystal $\mathcal{B}$ is isomorphic to $B(\infty)$. Once we have completed this step, it is not much harder to show $\mathcal{B}^\Lambda \cong B(\Lambda)$.

While the methods used in Section 6 differ from those of Grojnowski, the propositions and their proofs in Section 7 follow [17, Section 13] extremely closely.

7.1. Constructing the strict embedding Ψ

Recall Proposition 6.2 that said $\varepsilon_i^\vee(\widetilde{f}_j M) = \varepsilon_i^\vee(M)$ when $i \neq j$ but when $i = j$ either $\varepsilon_i^\vee(\widetilde{f}_i M) = \varepsilon_i^\vee(M)$ or $\varepsilon_i^\vee(M) + 1$.

Proposition 7.1. *Let M be a simple $R(\nu)$-module, and write $c = \varepsilon_i^\vee(M)$.*

(i) *Suppose $\varepsilon_i^\vee(\widetilde{f}_i M) = \varepsilon_i^\vee(M) + 1$. Then*

$$\widetilde{e}_i^\vee \widetilde{f}_i M \cong M. \tag{7.1}$$

(ii) *Suppose $\varepsilon_i^\vee(\widetilde{f}_j M) = \varepsilon_i^\vee(M)$ where i and j are not necessarily distinct. Then*

$$\left(\widetilde{e}_i{}^\vee\right)^c(\widetilde{f}_j M) \cong \widetilde{f}_j\left(\widetilde{e}_i{}^{\vee c} M\right). \tag{7.2}$$

Proof. For part (i), the Jump Lemma 6.5 gives us $\widetilde{f}_i M \cong \widetilde{f}_i{}^\vee M$. Therefore, $\widetilde{e}_i{}^\vee \widetilde{f}_i M \cong \widetilde{e}_i{}^\vee \widetilde{f}_i{}^\vee M \cong M$.

For part (ii) let $\overline{M} = (\widetilde{e}_i{}^\vee)^c M$ so that $\varepsilon_i^\vee(\overline{M}) = 0$ and we have a surjection $\operatorname{Ind} L(i^c) \boxtimes \overline{M} \twoheadrightarrow M$ as well as

$$\operatorname{Ind} L\left(i^c\right) \boxtimes \overline{M} \boxtimes L(j) \twoheadrightarrow \widetilde{f}_j M. \tag{7.3}$$

Note that as $c = \varepsilon_i^\vee(\widetilde{f}_j M)$, all composition factors of $(e_i^\vee)^c \widetilde{f}_j M$ are isomorphic to $(\widetilde{e}_i{}^\vee)^c \widetilde{f}_j M$, so there exists a surjection $(e_i^\vee)^c \widetilde{f}_j M \twoheadrightarrow (\widetilde{e}_i{}^\vee)^c \widetilde{f}_j M$. As $(e_i^\vee)^c$ is exact, we may apply it to (7.3) and compose with the map above yielding

$$\left(e_i^\vee\right)^c\left(\operatorname{Ind} L\left(i^c\right) \boxtimes \overline{M} \boxtimes L(j)\right) \twoheadrightarrow \left(\widetilde{e}_i{}^\vee\right)^c \widetilde{f}_j M. \tag{7.4}$$

In the case $j \neq i$, by the Mackey theorem [31, Proposition 2.8] $(e_i^\vee)^c(\operatorname{Ind} L(i^c) \boxtimes \overline{M} \boxtimes L(j))$ has a filtration whose subquotients are isomorphic to $\operatorname{Ind} \overline{M} \boxtimes L(j)$. So (7.4) yields a map

$$\operatorname{Ind} \overline{M} \boxtimes L(j) \twoheadrightarrow \left(\widetilde{e}_i{}^\vee\right)^c \widetilde{f}_j M, \tag{7.5}$$

which implies

$$\left(\widetilde{e}_i{}^\vee\right)^c \widetilde{f}_j M \cong \widetilde{f}_j \overline{M} \cong \widetilde{f}_j\left(\widetilde{e}_i{}^\vee\right)^c M. \tag{7.6}$$

In the case $j = i$, the subquotients are isomorphic to $\operatorname{Ind} \overline{M} \boxtimes L(i)$ or $\operatorname{Ind} L(i) \boxtimes \overline{M}$. But, by assumption $\varepsilon_i^\vee((\widetilde{e}_i{}^\vee)^c \widetilde{f}_i M) = 0$, so by Frobenius reciprocity we cannot have a nonzero map from $\operatorname{Ind} L(i) \boxtimes \overline{M}$ to $(\widetilde{e}_i{}^\vee)^c \widetilde{f}_i M$. As before, we must have

$$\operatorname{Ind} \overline{M} \boxtimes L(i) \twoheadrightarrow \left(\widetilde{e}_i{}^\vee\right)^c \widetilde{f}_i M \tag{7.7}$$

and so $(\widetilde{e}_i{}^\vee)^c \widetilde{f}_j M = (\widetilde{e}_i{}^\vee)^c \widetilde{f}_i M \cong \widetilde{f}_i \overline{M} = \widetilde{f}_i(\widetilde{e}_i{}^\vee)^c M = \widetilde{f}_j(\widetilde{e}_i{}^\vee)^c M$. $\square$

Proposition 7.2. *Let M be an irreducible $R(\nu)$-module, and write $c = \varepsilon_i^\vee(M)$, $\overline{M} = (\widetilde{e}_i{}^\vee)^c(M)$.*

(i) $\varepsilon_i(M) = \max\{\varepsilon_i(\overline{M}), c - \mathrm{wt}_i(\overline{M})\}$.
(ii) *Suppose $\varepsilon_i(M) > 0$. Then*

$$\varepsilon_i^\vee(\widetilde{e}_i M) = \begin{cases} c & \text{if } \varepsilon_i(\overline{M}) \geqslant c - \mathrm{wt}_i(\overline{M}), \\ c-1 & \text{if } \varepsilon_i(\overline{M}) < c - \mathrm{wt}_i(\overline{M}). \end{cases} \tag{7.8}$$

(iii) *Suppose $\varepsilon_i(M) > 0$. Then*

$$\left(\widetilde{e}_i{}^\vee\right)^{\varepsilon_i^\vee(\widetilde{e}_i M)}(\widetilde{e}_i M) = \begin{cases} \widetilde{e}_i(\overline{M}) & \text{if } \varepsilon_i(\overline{M}) \geqslant c - \mathrm{wt}_i(\overline{M}), \\ \overline{M} & \text{if } \varepsilon_i(\overline{M}) < c - \mathrm{wt}_i(\overline{M}). \end{cases} \tag{7.9}$$

Proof. Suppose $\varepsilon_i(M) > \varepsilon_i(\overline{M})$. Then $\text{jump}_i(M) = 0$ and by Proposition 6.7(v)

$$0 = \text{jump}_i(M) = \varepsilon_i(M) + \varepsilon_i^\vee(M) + \text{wt}_i(M) = \varepsilon_i(M) + c + \text{wt}_i(\overline{M}) - 2c \tag{7.10}$$

so that $\varepsilon_i(M) = c - \text{wt}_i(\overline{M})$, and clearly $\varepsilon_i(M) = \max\{\varepsilon_i(\overline{M}), c - \text{wt}_i(\overline{M})\}$. It is always the case that $\text{jump}_i(M) \geqslant 0$. If $\varepsilon_i(M) = \varepsilon_i(\overline{M})$, then as above $\varepsilon_i(M) = (c - \text{wt}_i(\overline{M})) + \text{jump}_i(M) \geqslant c - \text{wt}_i(\overline{M})$. So again $\varepsilon_i(M) = \max\{\varepsilon_i(\overline{M}), c - \text{wt}_i(\overline{M})\}$.

For part (ii) consider two cases.

Case 1 ($\varepsilon_i(\overline{M}) < c - \text{wt}_i(\overline{M})$): Recall by Proposition 6.7(v), $\text{jump}_i(\overline{M}) = \varepsilon_i^\vee(\overline{M}) + \varepsilon_i(\overline{M}) + \text{wt}_i(\overline{M}) = 0 + \varepsilon_i(\overline{M}) + \text{wt}_i(\overline{M})$ so $\text{jump}_i \overline{M} < c$ if and only if $\varepsilon_i(\overline{M}) < c - \text{wt}_i(\overline{M})$. Since $\text{jump}_i \overline{M} < c$ then $0 = \text{jump}_i((\widetilde{f_i}^\vee)^{c-1}\overline{M}) = \text{jump}_i(\widetilde{e_i}^\vee M)$ by (6.11). By the Jump Lemma 6.5, $\widetilde{f_i}(\widetilde{e_i}^\vee M) \cong \widetilde{f_i}^\vee(\widetilde{e_i}^\vee M) \cong M$. Hence $\widetilde{e_i}^\vee M = \widetilde{e_i} M$ and so $\varepsilon_i^\vee(\widetilde{e_i} M) = \varepsilon_i^\vee(\widetilde{e_i}^\vee M) = c - 1$.

Case 2 ($\varepsilon_i(\overline{M}) \geqslant c - \text{wt}_i(\overline{M})$): As above this case is equivalent to $\text{jump}_i \overline{M} \geqslant c$. Note if $c = 0$ then (ii) obviously holds by Proposition 6.2. If $c > 0$ by (6.11), we must have $0 < \text{jump}_i((\widetilde{f_i}^\vee)^{c-1}\overline{M}) = \text{jump}_i(\widetilde{e_i}^\vee M)$. Suppose that $\text{jump}_i(\widetilde{e_i} M) = 0$. Then as above $\widetilde{f_i}^\vee \widetilde{e_i} M \cong \widetilde{f_i}\widetilde{e_i} M \cong M$ and so $\widetilde{e_i} M \cong \widetilde{e_i}^\vee M$ yielding $\text{jump}_i(\widetilde{e_i}^\vee M) = 0$ which is a contradiction. So we must have $\text{jump}_i(\widetilde{e_i} M) > 0$. Then by the definition of jump_i, $\varepsilon_i^\vee(\widetilde{e_i} M) = \varepsilon_i^\vee(\widetilde{f_i}\widetilde{e_i} M) = \varepsilon_i^\vee(M) = c$.

For part (iii), first suppose $\varepsilon_i(\overline{M}) \geqslant c - \text{wt}_i(\overline{M})$. Then by part (ii) $\varepsilon_i^\vee(\widetilde{e_i} M) = c = \varepsilon_i(M)$. In other words $\varepsilon_i^\vee(\widetilde{e_i} M) = \varepsilon_i^\vee(\widetilde{f_i}\widetilde{e_i} M)$ so by Proposition 7.1 applied to $\widetilde{e_i} M$,

$$\widetilde{f_i}\left(\widetilde{e_i}^\vee\right)^c \widetilde{e_i} M \cong \left(\widetilde{e_i}^\vee\right)^c \widetilde{f_i}\widetilde{e_i} M \cong \left(\widetilde{e_i}^\vee\right)^c M = \overline{M}. \tag{7.11}$$

Hence $(\widetilde{e_i}^\vee)^c \widetilde{e_i} M \cong \widetilde{e_i}\overline{M}$.

Next suppose $\varepsilon_i(\overline{M}) < c - \text{wt}_i(\overline{M})$. Then by part (ii)

$$\varepsilon_i^\vee(\widetilde{e_i} M) = c - 1 = \varepsilon_i^\vee(M) - 1. \tag{7.12}$$

In other words $\varepsilon_i^\vee(\widetilde{f_i}\widetilde{e_i} M) = \varepsilon_i^\vee(\widetilde{e_i} M) + 1$, so by Proposition 7.1 applied to $\widetilde{e_i} M$,

$$\widetilde{e_i}^\vee M \cong \widetilde{e_i}^\vee \widetilde{f_i}\widetilde{e_i} M \cong \widetilde{e_i} M, \tag{7.13}$$

hence $(\widetilde{e_i}^\vee)^{c-1}\widetilde{e_i} M \cong (\widetilde{e_i}^\vee)^{c-1}\widetilde{e_i}^\vee M \cong (\widetilde{e_i}^\vee)^c M \cong \overline{M}$. $\square$

Proposition 7.3. *For each $i \in I$ define a map*

$$\begin{aligned} \Psi_i : \mathcal{B} &\to \mathcal{B} \otimes B_i, \\ M &\mapsto \left(\widetilde{e_i}^\vee\right)^c (M) \otimes b_i(-c), \end{aligned}$$

where $c = \varepsilon_i^\vee(M)$. Then Ψ_i is a strict embedding of crystals.

Proof. First we show that Ψ_i is a morphism of crystals. (M1) is obvious. For (M2) let $\overline{M} = (\widetilde{e_i}^\vee)^c M$. We compute

$$\begin{aligned}\mathrm{wt}\big(\psi_i(M)\big) &= \mathrm{wt}\big(\overline{M}\otimes b_i(-c)\big)\\ &= \mathrm{wt}(\overline{M}) + \mathrm{wt}\big(b_i(-c)\big)\\ &= \mathrm{wt}(M) + c\alpha_i - c\alpha_i = \mathrm{wt}(M).\end{aligned}\tag{7.14}$$

Consider first the case $j \neq i$. By Proposition 6.2

$$\begin{aligned}\varepsilon_j\big(\Psi_i(M)\big) &= \varepsilon_j\big(\overline{M}\otimes b_i(-c)\big)\\ &= \max\big\{\varepsilon_j(\overline{M}), \varepsilon_j\big(b_i(-c)\big) - \big\langle h_j, \mathrm{wt}(\overline{M})\big\rangle\big\}\\ &= \max\big\{\varepsilon_j(\overline{M}), -\infty\big\} = \varepsilon_j(\overline{M})\\ &= \varepsilon_j(M).\end{aligned}$$

In the case $j = i$, Proposition 7.2(i) implies

$$\begin{aligned}\varepsilon_i\big(\Psi_i(M)\big) &= \varepsilon_i\big(\overline{M}\otimes b_i(-c)\big)\\ &= \max\big\{\varepsilon_i(\overline{M}), \varepsilon_i\big(b_i(-c)\big) - \big\langle h_i, \mathrm{wt}(\overline{M})\big\rangle\big\} = \max\big\{\varepsilon_i(\overline{M}), c - \mathrm{wt}_i(\overline{M})\big\}\\ &= \varepsilon_i(M).\end{aligned}\tag{7.15}$$

Since for both crystals, $\varphi_j(b) = \varepsilon_j(b) + \langle h_j, \mathrm{wt}(b)\rangle$ it follows $\varphi_j(M) = \varphi_j(\Psi_i(M))$ for all $j \in I$.

It is clear that Ψ_i is injective. We will prove a stronger statement than (M3) and (M4), namely $\Psi_i(\widetilde{e_j}M) = \widetilde{e_j}(\Psi_i(M))$ and $\Psi_i(\widetilde{f_j}M) = \widetilde{f_j}(\Psi_i(M))$ which will show Ψ_i is not just a morphism of crystals, but since it is injective, Ψ_i is a strict embedding of crystals.

Observe

$$\widetilde{e_j}\big(\Psi_i(M)\big) = \widetilde{e_j}\big(\overline{M}\otimes b_i(-c)\big) = \begin{cases}\widetilde{e_j}\overline{M}\otimes b_i(-c) & \text{if } \varphi_j(\overline{M}) \geqslant \varepsilon_i(b_i(-c)) = c,\\ \overline{M}\otimes b_i(-c+1) & \text{if } \varphi_j(\overline{M}) < c.\end{cases}\tag{7.16}$$

We first consider the case when $j = i$. If $\varepsilon_i(M) = 0$, then clearly $\varepsilon_i(\overline{M}) = 0$ and further $\widetilde{e_i}M = \widetilde{e_i}\overline{M} = \mathbf{0}$. By Proposition 7.2(i)

$$\varepsilon_i(\overline{M}) = 0 = \varepsilon_i(M) = \max\big\{\varepsilon_i(\overline{M}), c - \mathrm{wt}_i(\overline{M})\big\} \geqslant c - \mathrm{wt}_i(\overline{M}),\tag{7.17}$$

yielding $\varphi_i(\overline{M}) = \varepsilon_i(\overline{M}) + \mathrm{wt}_i(\overline{M}) \geqslant (c - \mathrm{wt}_i(\overline{M})) + \mathrm{wt}_i(\overline{M}) = c$, so by (4.8), (4.10) we get

$$\widetilde{e_i}\Psi_i(M) = \widetilde{e_i}\overline{M}\otimes b_i(-c) = 0 = \Psi_i(0) = \Psi_i(\widetilde{e_i}M).\tag{7.18}$$

Now suppose $\varepsilon_i(M) > 0$. Using that $\varphi_i(\overline{M}) := \varepsilon_i(\overline{M}) + \mathrm{wt}_i(\overline{M})$, (4.8), and (4.10), Proposition 7.2 implies we can rewrite

$$\widetilde{e_i}\Psi_i(M) = \begin{cases}(\widetilde{e_i}^{\vee})^{c}\widetilde{e_i}M\otimes b_i(-c) & \text{if } \varepsilon_i(\overline{M}) \geqslant c - \mathrm{wt}_i(\overline{M})\\ (\widetilde{e_i}^{\vee})^{c-1}\widetilde{e_i}M\otimes b_i(-c+1) & \text{if } \varepsilon_i(\overline{M}) < c - \mathrm{wt}_i(\overline{M})\end{cases}\tag{7.19}$$

$$= \big(\widetilde{e_i}^{\vee}\big)^{\varepsilon_i^{\vee}(\widetilde{e_i}M)}\widetilde{e_i}M\otimes b_i\big(\varepsilon_i^{\vee}(\widetilde{e_i}M)\big)\tag{7.20}$$

$$= \Psi_i(\widetilde{e_i}M).\tag{7.21}$$

When $j \neq i$ note that $\varepsilon_i^\vee(\widetilde{e}_j M) = \varepsilon_i^\vee(M) = c$ as long as $\widetilde{e}_j M \neq \mathbf{0}$, by Proposition 6.2 applied to $\widetilde{e}_j M$. Part (ii) of Proposition 7.1 implies $\overline{M} = (\widetilde{e}_i^{\vee})^c M = \widetilde{f}_j (\widetilde{e}_i^{\vee})^c \widetilde{e}_j M$, so $\widetilde{e}_j \overline{M} = (\widetilde{e}_i^{\vee})^c \widetilde{e}_j M$. Therefore, by (7.16) as $\varepsilon_j(b_i(-c)) = -\infty$,

$$\widetilde{e}_j\big(\Psi_i(M)\big) = \widetilde{e}_j \overline{M} \otimes b_i(-c) = \big(\widetilde{e}_i^{\vee}\big)^c \widetilde{e}_j M \otimes b_i(-c) = \Psi_i(\widetilde{e}_j M). \tag{7.22}$$

In the case $\widetilde{e}_j M = \mathbf{0}$, Proposition 6.2 implies $\widetilde{e}_j \overline{M} = \mathbf{0}$ as well, so we compute

$$\widetilde{e}_j\big(\Psi_i(M)\big) = \widetilde{e}_j \overline{M} \otimes b_i(-c) = 0 = \Psi_i(0) = \Psi_i(\widetilde{e}_j M).$$

The proof that $\Psi_i(\widetilde{f}_j M) = \widetilde{f}_j(\Psi_i(M))$ is similar. □

7.2. Main theorems

In the following we use the characterization of $B(\infty)$ from Section 4.2 to implicitly prove $\mathcal{B}$ is isomorphic to $B(\infty)$.

Theorem 7.4. *The crystal $\mathcal{B}$ is isomorphic to $B(\infty)$.*

Proof. Recall that by abuse of notation, for irreducible modules M, we write $M \in \mathcal{B}$ as shorthand for $[M] \in \mathcal{B}$. We show that the crystal $\mathcal{B}$ satisfies the characterizing properties of $B(\infty)$ given in Proposition 4.3. Properties (B1)–(B4) are obvious with $\mathbb{1}$ the unique node with weight zero. The embedding $\Psi_i : \mathcal{B} \to \mathcal{B} \otimes B_i$ for (B5) was constructed in the previous section. (B6) follows from the definition of Ψ_i as $\varepsilon_j^\vee(M) \geqslant 0$ for all $M \in \mathcal{B}$, $j \in I$. For (B7) we must show that for $M \in \mathcal{B}$ other than $\mathbb{1}$, then there exists $i \in I$ such that $\Psi_i(M) = N \otimes \widetilde{f}_i^{\,n} b_i$ for some $N \in \mathcal{B}$ and $n > 0$. But every such M has $\varepsilon_i^\vee(M) > 0$ for at least one $i \in I$, so that N can be taken to be $\widetilde{e}_i^{\vee n}(M)$ for $n = \varepsilon_i^\vee(M) > 0$. □

Now we will show the data $(\mathcal{B}^\Lambda, \varepsilon_i^\Lambda, \varphi_i^\Lambda, \widetilde{e}_i^{\,\Lambda}, \widetilde{e}_i^{\,\Lambda}, \mathrm{wt}^\Lambda)$ of Section 5.3 defines a crystal graph and identify it as the highest weight crystal $B(\Lambda)$.

Theorem 7.5. *$\mathcal{B}^\Lambda$ is a crystal; furthermore the crystal $\mathcal{B}^\Lambda$ is isomorphic to $B(\Lambda)$.*

Proof. Proposition 8.2 of Kashiwara [28] gives us an embedding

$$\Upsilon^\infty : B(\Lambda) \to B(\infty) \otimes T_\Lambda \tag{7.23}$$

which identifies $B(\Lambda)$ as a subcrystal of $B(\infty) \otimes T_\Lambda$. The nodes of $B(\Lambda)$ are associated with the nodes of the image

$$\operatorname{Im} \Upsilon^\infty = \big\{ b \otimes t_\Lambda \;\big|\; \varepsilon_i^*(b) \leqslant \langle h_i, \Lambda \rangle, \text{ for all } i \in I \big\} \tag{7.24}$$

where $c = \varepsilon_i^*(b)$ is defined via $\Psi_i b = b' \otimes b_i(-c)$ for the strict embedding $\Psi_i : B(\infty) \to B(\infty) \otimes B_i$. The crystal data for $B(\Lambda)$ is thus inherited from that of $B(\infty) \otimes T_\Lambda$. Via our isomorphism $B(\infty) \otimes T_\Lambda \cong \mathcal{B} \otimes T_\Lambda$ of Theorem 7.4 and the description of

$$\Psi_i : \mathcal{B} \to \mathcal{B} \otimes B_i,$$
$$M \mapsto \left(\widetilde{e}_i^{\vee}\right)^{\varepsilon_i^{\vee}(M)} M \otimes b_i\left(-\varepsilon_i^{\vee}(M)\right) \tag{7.25}$$

the set

$$\left\{M \otimes t_\Lambda \in \mathcal{B} \otimes t_\Lambda \;\middle|\; \varepsilon_i^{\vee}(M) \leqslant \lambda_i,\ \text{for all } i \in I\right\} \tag{7.26}$$

endowed with the crystal data of $\mathcal{B} \otimes T_\Lambda$ is thus isomorphic to $B(\Lambda)$.

Recall from Section 5.3 this is precisely $\operatorname{Im} \Upsilon$, as $\varepsilon_i^{\vee}(M) \leqslant \lambda_i$ for all $i \in I$ if and only if $\mathrm{pr}_\Lambda M \neq \mathbf{0}$ which happens if and only if $M = \mathrm{infl}_\Lambda \mathcal{M}$ for some $\mathcal{M} \in \mathcal{B}^\Lambda$. By Kashiwara's proposition, we know $\operatorname{Im} \Upsilon \cong B(\Lambda)$ as crystals.

What remains is to check that the crystal data $\operatorname{Im} \Upsilon$ inherits from $\mathcal{B} \otimes T_\Lambda$ agrees with the data defined in Section 5.3 for $\mathcal{B}^\Lambda$. Once we verify this, we will have shown $\mathcal{B}^\Lambda$ is a crystal, $\mathcal{B}^\Lambda \cong B(\Lambda)$, and Υ is an embedding of crystals.

Let $\mathcal{M} \in \mathcal{B}^\Lambda$. Recall, since $\mathrm{pr}_\Lambda \mathrm{infl}_\Lambda \mathcal{M} \neq \mathbf{0}$, then $0 \leqslant \varphi_i^\Lambda(\mathrm{infl}_\Lambda \mathcal{M}) = \varphi_i^\Lambda(\mathcal{M})$ which was defined as $\max\{k \mid \mathrm{pr}_\Lambda \widetilde{f}_i^{\,k}(\mathrm{infl}_\Lambda \mathcal{M}) \neq \mathbf{0}\}$. We verify

$$\begin{aligned} \varphi_i(\Upsilon \mathcal{M}) &= \varphi_i(\mathrm{infl}_\Lambda \mathcal{M} \otimes t_\Lambda) \\ &= \varphi_i(\mathrm{infl}_\Lambda \mathcal{M}) + \lambda_i \\ &= \varepsilon_i(\mathrm{infl}_\Lambda \mathcal{M}) + \mathrm{wt}_i(\mathrm{infl}_\Lambda \mathcal{M}) + \lambda_i \\ &\overset{\text{Cor. 6.22}}{=} \varphi_i^\Lambda(\mathrm{infl}_\Lambda \mathcal{M}) = \varphi_i^\Lambda(\mathcal{M}). \end{aligned} \tag{7.27}$$

This computation, along with (5.11)–(5.14) completes the check that $(\mathcal{B}^\Lambda, \varepsilon_i^\Lambda, \varphi_i^\Lambda, \widetilde{e}_i^{\,\Lambda}, \widetilde{e}_i^{\,\Lambda}, \mathrm{wt}^\Lambda)$ is a crystal and isomorphic to $B(\Lambda)$. □

7.3. $\mathbf{U}_q^+$-module structures

Set

$$G_0^*(R) = \bigoplus_\nu G_0\big(R(\nu)\big)^*, \qquad G_0^*\big(R^\Lambda\big) = \bigoplus_\nu G_0\big(R^\Lambda(\nu)\big)^*$$

where, by V^* we mean the restricted linear dual $\mathrm{Hom}_{\mathcal{A}}(V, \mathcal{A})$. Because $G_0(R)$ and $G_0(R^\Lambda)$ are ${}_{\mathcal{A}}\mathbf{U}_q^+$-modules, we can endow $G_0^*(R)$, $G_0^*(R^\Lambda)$ with a left ${}_{\mathcal{A}}\mathbf{U}_q^+$-module structure in several ways, via a choice of anti-automorphism. Here we denote by $*$ the $\mathcal{A}$-linear anti-automorphism defined by

$$e_i^* = e_i \quad \text{for all } i \in I.$$

Specifically, for $y \in {}_{\mathcal{A}}\mathbf{U}_q^+$, $\gamma \in G_0^*(R)$ or $G_0^*(R^\Lambda)$, and N simple, set

$$(y \cdot \gamma)\big([N]\big) = \gamma\big(y^*[N]\big)$$

where we will identify e_i^Λ with e_i.

$G_0(R(\nu))^*$ has basis given by $\{\delta_M \mid M \in \mathcal{B},\ \mathrm{wt}(M) = -\nu\}$ defined by

$$\delta_M([N]) = \begin{cases} q^{-r}, & M \cong N\{r\}, \\ 0, & \text{otherwise}, \end{cases}$$

where N ranges over simple $R(\nu)$-modules. We set $\mathrm{wt}(\delta_M) = -\mathrm{wt}(M)$. Likewise $G_0(R^\Lambda(\nu))^*$ has basis $\{\mathfrak{d}_{\mathcal{M}} \mid \mathcal{M} \in \mathcal{B}^\Lambda,\ \mathrm{wt}(\mathcal{M}) = -\nu + \Lambda\}$ defined similarly. Note that if δ_M has degree d then $\delta_{M\{1\}} = q^{-1}\delta_M$ has degree $d-1$. Recall $\mathbb{1} \in \mathcal{B}$ denotes the trivial $R(0)$-module and we will also write $\mathbb{1} \in \mathcal{B}^\Lambda$ for the trivial $R^\Lambda(0)$-module.

Lemma 7.6.

(i) $e_i^{(m)} \cdot \delta_{\mathbb{1}} = \delta_{L(i^m)} \in G_0(R(mi))^*$; $e_i^{(m)} \cdot \mathfrak{d}_{\mathbb{1}} = 0 \in G_0(R^\Lambda(mi))^* \subseteq G_0^*(R^\Lambda)$ *if* $m \geqslant \lambda_i + 1$.
(ii) $G_0^*(R)$ *is generated by* $\delta_{\mathbb{1}}$ *as a* ${}_{\mathcal{A}}\mathbf{U}_q^+$*-module;* $G_0^*(R^\Lambda)$ *is generated by* $\mathfrak{d}_{\mathbb{1}}$ *as a* ${}_{\mathcal{A}}\mathbf{U}_q^+$*-module.*

Proof. The first part follows since $e_i^{(m)} L(i^m) \cong \mathbb{1}$ and the only irreducible module N for which $e_i^{(m)} N$ is a nonzero $R(0)$-module is $N \cong L(i^m)\{r\}$ for some $r \in \mathbb{Z}$. Recall $\mathrm{pr}_\Lambda L(i^m) = \mathbf{0}$ if and only if $m \geqslant \lambda_i + 1$.

For the second part, recall $\mathbb{1}$ co-generates $G_0(R)$ (resp. $G_0(R^\Lambda)$) in the sense that for any irreducible M, there exist $i_t \in I$ such that

$$e_{i_k}^{(m_k)} \dots e_{i_2}^{(m_2)} e_{i_1}^{(m_1)} M \cong \mathfrak{a}\mathbb{1},$$

where $m_t = \varepsilon_{i_t}(\widetilde{e_{i_{t-1}}}^{m_{t-1}} \dots \widetilde{e_{i_1}}^{m_1} M)$ and $\mathfrak{a} \in \mathcal{A}$ (in fact $\mathfrak{a} = q^r$ for some $r \in \mathbb{Z}$). So certainly $\delta_{\mathbb{1}}$ generates $G_0^*(R)$ (resp. $\mathfrak{d}_{\mathbb{1}}$ generates $G_0(R^\Lambda)$).

More specifically, an inductive argument relying on "triangularity" with respect to ε_i gives $\delta_M \in {}_{\mathcal{A}}\mathbf{U}_q^+ \cdot \delta_{\mathbb{1}}$ and $\mathfrak{d}_{\mathcal{M}} \in {}_{\mathcal{A}}\mathbf{U}_q^+ \cdot \mathfrak{d}_{\mathbb{1}}$. □

Lemma 7.7.

(i) *The maps*

$$ {}_{\mathcal{A}}\mathbf{U}_q^+ \xrightarrow{F} G_0^*(R), \qquad {}_{\mathcal{A}}\mathbf{U}_q^+ \xrightarrow{\mathcal{F}} G_0^*(R^\Lambda), \tag{7.28}$$

$$ y \mapsto y \cdot \delta_{\mathbb{1}}, \qquad y \mapsto y \cdot \mathfrak{d}_{\mathbb{1}} \tag{7.29}$$

are ${}_{\mathcal{A}}\mathbf{U}_q^+$*-module homomorphisms.*
(ii) F *and* $\mathcal{F}$ *are surjective.*
(iii) $\ker \mathcal{F} \ni e_i^{(\lambda_i+1)}$ *for all* $i \in I$.

Proof. To show F, $\mathcal{F}$ are ${}_{\mathcal{A}}\mathbf{U}_q^+$-maps, we need only to check the Serre relations (6.16) vanish on $G_0^*(R)$, $G_0^*(R^\Lambda)$. But as the corresponding operators are invariant under $*$ and vanish on any $[N]$, they certainly kill any δ_M, $\mathfrak{d}_{\mathcal{M}}$.

Now F (resp. $\mathcal{F}$) is clearly surjective as it contains the generator $\delta_{\mathbb{1}}$ (resp. $\mathfrak{d}_{\mathbb{1}}$) in its image.

The third statement follows from part (i) of Lemma 7.6. □

If $V(\Lambda)$ is the irreducible highest weight $\mathbf{U}_q(\mathfrak{g})$-module with highest weight Λ and highest weight vector v_Λ then its $\mathcal{A}$-form, or integral form, ${}_{\mathcal{A}}V(\Lambda)$ is the $U_{\mathcal{A}}$-submodule of $V(\Lambda)$ generated by v_Λ. In particular, ${}_{\mathcal{A}}V(\Lambda) = {}_{\mathcal{A}}\mathbf{U}_q^- \cdot v_\Lambda$. We let $V(\Lambda)^*$ denote the graded dual of $V(\Lambda)$, whose elements are sums of δ_v, $v \in V(\Lambda)$. If $v \in V(\Lambda)$ has weight μ then $\delta_v \in V(\Lambda)^*$ has weight $-\mu$ and $e_i v$, if nonzero, has weight $\mu + i$ in the notation of this paper. We set

$$ {}_{\mathcal{A}}V^*(\Lambda) = {}_{\mathcal{A}}\mathbf{U}_q^+ \cdot \delta_{v_\Lambda} $$

endowed with the left ${}_{\mathcal{A}}\mathbf{U}_q^+$-module structure

$$ y \cdot \delta_v(w) = \delta_v\bigl(y^* w\bigr). $$

Note that the $-\mu$ weight space of the dual is the dual of the μ weight space, and that both weight spaces are free $\mathcal{A}$-modules of finite rank.

As a left ${}_{\mathcal{A}}\mathbf{U}_q^+$-module

$$ {}_{\mathcal{A}}V^*(\Lambda) \cong {}_{\mathcal{A}}\mathbf{U}_q^+ / \sum_{i \in I} {}_{\mathcal{A}}\mathbf{U}_q^+ \cdot e_i^{(\lambda_i+1)}. \tag{7.30} $$

We emphasize that parts 2 and 3 of the theorem below are new and settle part of the cyclotomic quotient conjecture in arbitrary type. While part 1 follows from [33, Theorem 8], here we have given a new proof of it modeled after the work of Grojnowski [17] using crystals to verify the rank of $G_0(R(\nu))$.

Theorem 7.8. *As ${}_{\mathcal{A}}\mathbf{U}_q^+$ modules*

1. ${}_{\mathcal{A}}\mathbf{U}_q^+ \cong G_0^*(R)$,
2. ${}_{\mathcal{A}}V^*(\Lambda) \cong G_0^*(R^\Lambda)$,
3. ${}_{\mathcal{A}}V(\Lambda) \cong G_0(R^\Lambda)$.

Proof. Note that both F and $\mathcal{F}$ are surjective and preserve weight in the sense that $\mathrm{wt}(e_i) = i$ in the notation of this paper. We know the dimension of the ν-weight space of $\mathbf{U}_q^+$ is

$$ \begin{aligned} \bigl|\bigl\{b \in B(\infty) \bigm| \mathrm{wt}(b) = -\nu\bigr\}\bigr| &= \bigl|\bigl\{M \in \mathcal{B} \bigm| \mathrm{wt}(M) = -\nu\bigr\}\bigr| \\ &= \mathrm{rank}_{\mathcal{A}}\, G_0\bigl(R(\nu)\bigr) \\ &= \mathrm{rank}_{\mathcal{A}}\, G_0\bigl(R(\nu)\bigr)^*. \end{aligned} $$

Because $\mathcal{A}$ is an integral domain, a surjection between two free $\mathcal{A}$-modules of the same (finite) rank must be an injection. Hence F must also be injective and hence an isomorphism.

Since the left ideal $\sum_{i \in I} {}_{\mathcal{A}}\mathbf{U}_q^+ \cdot e_i^{(\lambda_i+1)}$ is contained in the kernel of $\mathcal{F}$ by part (iii) of Lemma 7.7, $\mathcal{F}$ induces a surjection

$$ {}_{\mathcal{A}}V^*(\Lambda) \twoheadrightarrow G_0^*\bigl(R^\Lambda\bigr). $$

The dimension of the $-\Lambda+\nu$ weight space of $V(\Lambda)^*$ is the same as

$$\dim V(\Lambda)_{\Lambda-\nu} = \bigl|\bigl\{\mathfrak{b} \in B(\Lambda) \bigm| \mathrm{wt}(\mathfrak{b}) = \Lambda-\nu\bigr\}\bigr| = \bigl|\bigl\{\mathcal{M} \in \mathcal{B}^{\Lambda} \bigm| \mathrm{wt}(\mathcal{M}) = \Lambda-\nu\bigr\}\bigr| \tag{7.31}$$

$$= \mathrm{rank}_{\mathcal{A}}\, G_0\bigl(R^{\Lambda}(\nu)\bigr) = \mathrm{rank}_{\mathcal{A}}\, G_0\bigl(R^{\Lambda}(\nu)\bigr)^*, \tag{7.32}$$

so as above, $\mathcal{F}$ must in fact be an isomorphism.

The third statement follows from dualizing with respect to the anti-automorphism $*$. □

We note that [31] proves a stronger statement than part 1 of Theorem 7.8, namely that ${}_{\mathcal{A}}\mathbf{f} \cong K_0(R)$ as $\mathcal{A}$-bialgebras. So in particular, as ${}_{\mathcal{A}}\mathbf{U}_q^+$-modules, ${}_{\mathcal{A}}\mathbf{U}_q^+ \cong K_0(R)$. Using their result yields another proof that ${}_{\mathcal{A}}\mathbf{U}_q^+ \cong G_0(R)$ as ${}_{\mathcal{A}}\mathbf{U}_q^+$-modules.

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