



Higher integrality conditions, volumes and Ehrhart polynomials

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Received 6 December 2009; accepted 18 October 2010

Available online 4 November 2010

Communicated by Gil Kalai

Abstract

A polytope is integral if all of its vertices are lattice points. The constant term of the Ehrhart polynomial of an integral polytope is known to be 1. In previous work, we showed that the coefficients of the Ehrhart polynomial of a lattice-face polytope are volumes of projections of the polytope. We generalize both results by introducing a notion of k -integral polytopes, where 0-integral is equivalent to integral. We show that the Ehrhart polynomial of a k -integral polytope P has the properties that the coefficients in degrees less than or equal to k are determined by a projection of P , and the coefficients in higher degrees are determined by slices of P . A key step of the proof is that under certain generality conditions, the volume of a polytope is equal to the sum of volumes of slices of the polytope.

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MSC: primary 52B20; secondary 52A38, 05A15

Keywords: k -Integral; Volume; Ehrhart polynomial

1. Introduction

Let V be a finite-dimensional real vector space. A subset $\Lambda \subset V$ is called a *lattice of V* if Λ is a discrete additive subgroup of V and spans V . Let $\bar{\Lambda}$ be the rational vector space generated by Λ . Any point in Λ is called a *lattice point*, and any point in $\bar{\Lambda}$ is called a *rational point*.

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¹ The author is partially supported by the National Security Agency under Grant No. H98230-09-1-0029.

A subset $\Gamma \subset \Lambda$ is a *sublattice of Λ* if Γ is a subgroup of Λ . We say a sublattice Γ is of *rank r* if $\text{aff}(\Gamma)$ has dimension r .

We assume familiarity with basic definitions of polytopes as presented in [14]. An *integral polytope* is a convex polytope whose vertices are all lattice points. A *rational polytope* is a convex polytope whose vertices are all rational points.

For any sublattice $\Gamma \subset \Lambda$, let U be the subspace of V spanned by Γ . For any polytope P lying in an affine space that is a translation of U , we define the *volume of P normalized to the lattice Γ* to be the integral

$$\text{Vol}_\Gamma(P) := \int_P 1 \, d\Gamma, \tag{1.1}$$

where $d\Gamma$ is the canonical Lebesgue measure defined by the lattice Γ .

For any point $\mathbf{y} \in V$ and any set $S \subset V$, we denote by $S + \mathbf{y}$ the set $\{\mathbf{s} + \mathbf{y} \mid \mathbf{s} \in S\}$ obtained from translating S by \mathbf{y} . For any affine space $W \subset V$, we denote by $\text{lin}(W)$ the translation of W to the origin. It is easy to see that $W = \text{lin}(W) + \mathbf{y}$, for some (equivalently all) \mathbf{y} in W .

If P is a rational polytope in V , we denote by $\text{lin}(P)$ the translation of $\text{aff}(P)$ to the origin and denote by $\Lambda_{\text{lin}(P)}$ the lattice $\text{lin}(P) \cap \Lambda$. One checks $\Lambda_{\text{lin}(P)}$ spans $\text{lin}(P)$. Thus, we can define $\text{Vol}_{\Lambda_{\text{lin}(P)}}(P)$, which we often refer as the *normalized volume of P* .

Throughout the paper, we will assume V has dimension D (so Λ is of rank D) and fix a basis $\mathbf{e} = (\mathbf{e}^1, \mathbf{e}^2, \dots, \mathbf{e}^D)$ of the lattice Λ . Then we can represent each point in V by its coordinates with respect to \mathbf{e} , namely, for any point $\sum_{i=1}^D c_i \mathbf{e}^i \in V$, we say its *coordinates (with respect to \mathbf{e})* are (c_1, \dots, c_D) . This gives us an isomorphism between V and \mathbb{R}^D , and Λ and \mathbb{Z}^D . For convenience, we denote by $\Lambda_k := \langle \mathbf{e}^1, \dots, \mathbf{e}^k \rangle$ the lattice generated by $\mathbf{e}^1, \dots, \mathbf{e}^k$, $\Lambda^k := \langle \mathbf{e}^{k+1}, \dots, \mathbf{e}^D \rangle$ the lattice generated by $\mathbf{e}^{k+1}, \dots, \mathbf{e}^D$, V_k the k -dimensional subspace of V spanned by $\mathbf{e}^1, \dots, \mathbf{e}^k$, and V^k the $(d - k)$ -dimensional subspace of V spanned by $\mathbf{e}^{k+1}, \dots, \mathbf{e}^D$. Let $\pi^{(k)} : V \rightarrow V_{D-k}$ be the projection that maps $\sum_{i=1}^D c_i \mathbf{e}^i$ to $\sum_{i=1}^{D-k} c_i \mathbf{e}^i$, or equivalently the map that forgets the last k coordinates.

For any polytope P in V , we denote by

$$i(P) := \#(P \cap \Lambda)$$

the number of lattice points inside P . Furthermore, for positive integer $m \in \mathbb{N}$, we denote by $i(P, m)$ the number of lattice points in mP , where $mP = \{mx \mid x \in P\}$ is the *m th dilation of P* . Eugène Ehrhart [7] discovered that for any d -dimensional integral polytope, $i(P, m)$ is a polynomial of degree d in m . Thus, we call $i(P, m)$ the *Ehrhart polynomial* of P . Ehrhart’s theory was developed in the 1960s, and there has been much work on the coefficients of Ehrhart polynomials since then. It is well known that for any integral polytope P , the leading, second and last coefficients of $i(P, m)$ are the normalized volume of P , one half of the normalized volume of the boundary of P and 1, respectively. Despite the work in [12,13,5] (which will be discussed in Section 9), we still do not have a very good understanding of when the coefficients of Ehrhart polynomials have simple geometric descriptions. In [8], the author showed that for any d -dimensional integral cyclic polytope P , we have that

$$i(P, m) = \text{Vol}_\Lambda(mP) + i(\pi(P), m) = \sum_{j=0}^D \text{Vol}_{\Lambda_j}(\pi^{(D-j)}(P))m^j, \tag{1.2}$$

where $D = d$. In [9,10], the author generalized the family of integral cyclic polytope to a bigger family of integral polytopes, *lattice-face* polytopes, and showed that their Ehrhart polynomials also satisfy (1.2).

The motivation of this paper is to prove a conjecture given in [9, Conjecture 8.5], which can be viewed as generalizations of both the theorem for lattice-face polytopes shown in [9,10] and the fact that the constant term of the Ehrhart polynomial of an integral polytope is 1. In fact, we will prove a stronger version of this conjecture. We need the following definitions before we can state the result.

Definition 1.1. Any affine space U in V is *integral with respect to ϵ* if

$$\pi^{(D-\dim(U))}(U \cap \Lambda) = \Lambda_{\dim(U)}.$$

A polytope is *affinely integral with respect to ϵ* if its affine hull is integral with respect to ϵ .

Suppose $0 \leq k \leq d \leq D$. A d -dimensional polytope in V is *k -integral with respect to ϵ* if any face of P of dimension less than or equal to k is affinely integral with respect to ϵ .

In particular, when $k = d$, we call P a *fully integral* polytope with respect to ϵ .

Note that the definition of 0-integral polytopes is the same as that of integral polytopes. Thus, any k -integral polytope is an integral polytope. Since we fix the lattice basis ϵ in the paper, we often omit “with respect to ϵ ” unless it is not clear from the context.

Definition 1.2. Let S be a set in V and $0 \leq k \leq d$. For any $\mathbf{y} \in \pi^{(D-k)}(V)$, we define $\pi_{D-k}(\mathbf{y}, S)$ to be the intersection of S with the inverse image of \mathbf{y} under $\pi^{(D-k)}$, and we call it *the slice of S over \mathbf{y}* .

Our first main theorem is the following.

Theorem 1.3. Suppose $0 \leq k \leq d \leq D$ and $P \subset V$ is a k -integral d -dimensional polytope. Then the Ehrhart polynomial of P is given by

$$\begin{aligned} i(P, m) &= m^k \left(\sum_{\mathbf{y} \in \pi^{(D-k)}(P) \cap \Lambda} i(\pi_{D-k}(\mathbf{y}, P), m) - 1 \right) + i(\pi^{(D-k)}(P), m) \\ &= m^k \left(\sum_{\mathbf{y} \in \pi^{(D-k)}(P) \cap \Lambda} i(\pi_{D-k}(\mathbf{y}, P), m) - 1 \right) + \sum_{j=0}^k \text{Vol}_{\Lambda_j}(\pi^{(D-j)}(P)) m^j. \end{aligned}$$

We will see in Corollary 7.10 that each $\pi_{D-k}(\mathbf{y}, P)$ appearing in the above formula is an integral polytope, so the constant term of $i(\pi_{D-k}(\mathbf{y}, P), m)$ is 1. Therefore, Theorem 1.3 says that the coefficient of m^j in the Ehrhart polynomial of a k -integral d -dimensional polytope P is $\text{Vol}_{\Lambda_j}(\pi^{(D-j)}(P))$ for $0 \leq j \leq k$, and is the coefficient of m^{j-k} in the sum of the Ehrhart polynomials of the slices of P over lattice points in $\pi^{(D-k)}(P)$ for $k + 1 \leq j \leq d$.

One consequence of Theorem 1.3 is that the coefficients of $i(P, m)$ are all positive for any d -dimensional polytope that is $(d - 2)$ -integral (with respect to some lattice basis). This provides

one possible way to prove positivity conjectures on coefficients of Ehrhart polynomials of special families of polytopes. It was recently conjectured that the coefficients of the Ehrhart polynomials of both Birkhoff polytopes and matroid polytopes are positive. One could approach these conjectures by proving these polytopes are $(d - 2)$ -integral, or by showing there exist subdivisions of these polytopes into $(d - 2)$ -integral polytopes (not necessarily with respect to a fixed lattice basis) satisfying certain conditions.

Another possible application of Theorem 1.3 is that it provides a different approach to calculating coefficients of Ehrhart polynomials. Most known algorithms for calculating the coefficients of Ehrhart polynomials are more efficient for obtaining the higher degree coefficients. Since we know that the constant term is always 1, it is the slowest to get the linear term coefficient. However, if a polytope is 1-integral, the coefficient of the linear term in its Ehrhart polynomial is just the volume of an interval, which is very easy to calculate. Although not all integral polytopes are 1-integral, we might be able to find an algorithm based on Theorem 1.3 or the idea of its proof that can obtain lower degree coefficients efficiently.

Recall that the leading coefficient of the Ehrhart polynomial of a polytope is the normalized volume of the polytope. If we only look at the leading coefficients in Theorem 1.3, we get that the normalized volume of P is equal to the sum of the normalized volume of slices of P over lattice points in $\pi^{(D-k)}(P)$, provided that P is k -integral. In fact, we can relax the condition of k -integral by introducing the following definition.

Definition 1.4. Any affine space U in V is in general position with respect to ϵ if

$$\pi^{(D-\dim(U))}(U) = V_{\dim(U)}.$$

A polytope is in affinely general position with respect to ϵ if its affine hull is in general position with respect to ϵ .

Suppose $0 \leq k \leq d \leq D$. A d -dimensional polytope in V is in k -general position with respect to ϵ if any face of P of dimension less than or equal to k is in affinely general position with respect to ϵ .

In particular, when $k = d$, we say P is in fully general position with respect to ϵ .

Again, we often omit “with respect to ϵ ” if there is no possibility of confusion.

Our result on volumes and slices is the following.

Theorem 1.5. Suppose $0 < k < d \leq D$ and $P \subset V$ is a d -dimensional $(k - 1)$ -integral polytope in k -general position. Let $\Lambda_P := \text{aff}(P) \cap \Lambda$. Then the normalized volume of P is given by

$$\text{Vol}_{\text{lin}(P)}(P) = \sum_{\mathbf{y} \in \pi^{(D-k)}(\Lambda_P)} \text{Vol}_{\Lambda^k \cap \Lambda_{\text{lin}(P)}}(\pi_{D-k}(\mathbf{y}, P)). \tag{1.3}$$

We remark that Theorem 1.5 is trivially true when $k = 0$ because the right-hand side of (1.3) becomes $\text{Vol}_{\text{lin}(P)}(P)$. However, the theorem does not hold for $k = d$. The most interesting case of Theorem 1.5 is when $k = 1$. If $k = 1$, the condition becomes that P is an integral polytope and is in 1-general position. However, for any integral polytope, it is always in 1-general position with respect to some basis. Hence, formula (1.3) with $k = 1$ can be applied to any integral polytope. Moreover, any rational polytope can be dilated to an integral polytope, so one can obtain a formula modified from (1.3) with $k = 1$ to calculate volumes of rational polytopes.

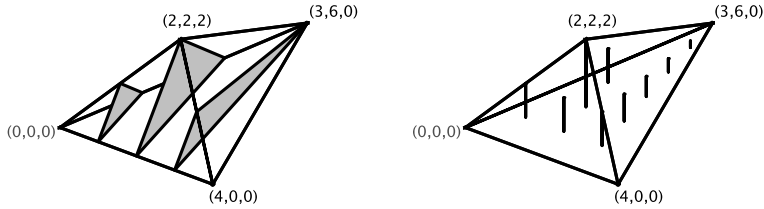


Fig. 1. Slices of a polytope.

Although Theorem 1.5 can be viewed as a corollary to Theorem 1.3, it actually serves as a key step in proving Theorem 1.3. In fact, most of the paper will be dedicated to the proof of Theorem 1.5.

The plan of this paper is as follows: In Section 2, we show examples of our main theorems. In Section 3, we introduce basic definitions and lemmas and reduce Theorem 1.5 to Proposition 3.16.

In Sections 4, 5 and 6, we prove a special case of Proposition 3.16 when $k = 1$. We first reduce the problem to the full-dimensional case, and then give two proofs: one uses a result of Barvinok [2] suggested by the referee, and the other (which we only sketch) carries out the calculation directly using techniques developed in [9]. In Section 7, we discuss the properties of slices and projections of a polytope. Using these, we complete the proof of Proposition 3.16 and thus the proof of Theorem 1.5 in Section 8. Finally, in Section 9, we prove Theorem 1.3 by using Theorem 1.5 and a local formula relating the number of lattice points to volumes of faces for integral polytopes [11–13,5].

2. Examples of the theorems

In this section, we give examples of the main theorems and also show that the conditions in Theorem 1.5 are in fact necessary. Throughout the section, we assume that $D = d$, $\Lambda = \mathbb{Z}^d$ with the standard basis, and $V = \mathbb{R}^d$.

Example 2.1 (Example of Theorem 1.3). Consider the 3-dimensional polytope

$$P = \text{conv}\{(0, 0, 0), (4, 0, 0), (3, 6, 0), (2, 2, 2)\} \subset \mathbb{R}^3.$$

One checks that P is 1-integral.

Clearly $\pi^{(2)}(P) = [0, 4]$, so the lattice points in $\pi^{(2)}(P)$ are 0, 1, 2, 3 and 4. In the picture on the left side of Fig. 1, the three shaded triangles are the slices of P over the lattice points 1, 2 and 3. The other two slices of P over lattice points are the single points $\pi_2(0, P) = (0, 0, 0)$ and $\pi_2(4, P) = (4, 0, 0)$. We calculate the Ehrhart polynomial of each slice and get $i(\pi_2(0, P), m) = 1$, $i(\pi_2(1, P), m) = m^2 + 2m + 1$, $i(\pi_2(2, P), m) = 4m^2 + 4m + 1$, $i(\pi_2(3, P), m) = 3m^2 + 4m + 1$ and $i(\pi_2(4, P), m) = 1$. Their sum is

$$8m^2 + 10m + 5.$$

We also have that

$$\text{Vol}_{\Lambda_1}(\pi^{(2)}(P)) = 4, \quad \text{and} \quad \text{Vol}_{\Lambda_0}(\pi^{(3)}(P)) = 1.$$

Therefore, applying Theorem 1.3 with $k = 1$, we conclude that 8, 10, 4 and 1 are the coefficients for m^3, m^2, m^1 and m^0 in $i(P, m)$. Hence, the Ehrhart polynomial of P is given by

$$i(P, m) = 8m^3 + 10m^2 + 4m + 1.$$

The rest of the examples in this section will be related to Theorem 1.5. Since all the polytopes we will consider are full-dimensional, we have that $\Lambda_{\text{lin}(P)} = \Lambda_P = \Lambda, \pi^{(D-k)}(\Lambda_P) = \Lambda_k$ and $\Lambda^k \cap \Lambda_{\text{lin}(P)} = \Lambda^k$. Hence, (1.3) can be simplified to

$$\text{Vol}_\Lambda(P) = \sum_{\mathbf{y} \in \Lambda_{d-k}} \text{Vol}_{\Lambda^k}(\pi_{d-k}(\mathbf{y}, P)). \tag{2.1}$$

Example 2.2 (Example of Theorem 1.5). Let P be the same polytope as in Example 2.1. Although P is not 2-integral, it is 1-integral and in 2-general position. We will apply Theorem 1.5 on P for $k = 2$. The volume of P (normalized to Λ) is 8. We will show the right-hand side of (1.3), or equivalently the right-hand side of (2.1), gives 8 as well.

We have that $\pi^{(1)}(P) = \text{conv}\{(0, 0), (4, 0), (3, 6)\}$. There are 9 lattice points in the interior of $\pi^{(1)}(P)$ and 8 lattice points on the boundary. In the picture on the right side of Fig. 1, the 9 line segments are the slices of P over the lattice points in the interior of $\pi^{(1)}(P)$. For example, the left-most line segment is the slice of P over $(1, 1)$:

$$\pi_1((1, 1), P) = \text{conv}\{(1, 1, 0), (1, 1, 1)\}.$$

The slices over the boundary lattice points are all single points, thus have 0 Λ^2 -volume. Hence, we only calculate the Λ^2 -volume of the slices over interior lattice points:

$$\begin{aligned} \text{Vol}_{\Lambda^2}(\pi_1((1, 1), P)) &= 1, & \text{Vol}_{\Lambda^2}(\pi_1((2, 1), P)) &= 1, & \text{Vol}_{\Lambda^2}(\pi_1((2, 2), P)) &= 2, \\ \text{Vol}_{\Lambda^2}(\pi_1((2, 3), P)) &= 1, & \text{Vol}_{\Lambda^2}(\pi_1((3, 1), P)) &= 1, & \text{Vol}_{\Lambda^2}(\pi_1((3, 2), P)) &= \frac{4}{5}, \\ \text{Vol}_{\Lambda^2}(\pi_1((3, 3), P)) &= \frac{3}{5}, & \text{Vol}_{\Lambda^2}(\pi_1((3, 4), P)) &= \frac{2}{5}, & \text{Vol}_{\Lambda^2}(\pi_1((3, 5), P)) &= \frac{1}{5}. \end{aligned}$$

The sum is 8, as desired.

The hypothesis in Theorem 1.5 is that P is $(k - 1)$ -integral and in k -general position. One might wonder whether it is possible to relax the condition and still get the same result (1.3). There are two natural ways to relax the condition: (1) only requiring P to be $(k - 1)$ -integral; (2) for $k \geq 2$, requiring P to be $(k - 2)$ -integral and in k -general position. In the next two examples, we show that (1.3) or equivalently (2.1) does not hold if we relax the hypothesis in Theorem 1.5 to either (1) or (2).

Example 2.3. Consider P to be the unit square in \mathbb{R}^2 . Then P is integral, but not in 1-general position, because it has edges $e_1 = \text{conv}\{(0, 0), (0, 1)\}$ and $e_2 = \text{conv}\{(1, 0), (1, 1)\}$ that are not in affinely general position.

We have that $\text{Vol}_\Lambda(P) = 1$ but $\sum_{\mathbf{y} \in \Lambda_1} \text{Vol}_{\Lambda^1}(\pi_1(\mathbf{y}, P)) = \text{Vol}_{\Lambda^1}(e_1) + \text{Vol}_{\Lambda^1}(e_2) = 2$.

Therefore, Theorem 1.5 does not hold for $(k - 1)$ -integral polytopes that are not in k -general position.

Example 2.4. Consider the polytope

$$P = \text{conv}\{(0, 0, 0), (4, 0, 0), (3, 3, 0), (2, 1, 5)\}.$$

We check that P is an integral polytope that is in 2-general position. Since the edges $\text{conv}\{(0, 0, 0), (2, 1, 5)\}$ and $\text{conv}\{(4, 0, 0), (2, 1, 5)\}$ are not affinely integral, P is not 1-integral.

We calculate $\text{Vol}_\Lambda(P) = 10$, but $\sum_{\mathbf{y} \in \Lambda_2} \text{Vol}_{\Lambda^2}(\pi_1(\mathbf{y}, P)) = 8$.

We thus conclude that Theorem 1.5 does not hold for $(k - 2)$ -integral polytopes that are in k -general position.

3. Preliminaries

In this section, we investigate the basic properties of the conditions and formulas involved in our main theorems. The lemmas and definitions introduced here will be used as the foundation of this paper.

First, we see that any k -integral polytope P is integral. Hence, $\text{aff}(P)$ contains a lattice point, say β . If we let $Q := P - \beta$, it is easy to check that to prove Theorem 1.3 and Theorem 1.5 for P is equivalent to proving them for the polytope Q . Note that Q has the property that $\text{aff}(Q)$ contains the origin, thus is a subspace of V . Therefore, we give the following definition:

Definition 3.1. For any polytope $P \subset V$, we say P is *central* if $\text{aff}(P)$ contains the origin, or equivalently, $\text{aff}(P) = \text{lin}(P)$ is a subspace of V .

Based on above discussion, we conclude:

Lemma 3.2. *It is sufficient to prove Theorem 1.3 and Theorem 1.5 with the assumption that P is central.*

Lemma 3.2 is helpful for proving Theorem 1.5 in particular. If P is central, then $\Lambda_P = \Lambda_{\text{lin}(P)}$. So formula (1.3) can be simplified to

$$\text{Vol}_{\Lambda_P}(P) = \sum_{\mathbf{y} \in \pi^{(D-k)}(\Lambda_P)} \text{Vol}_{\Lambda^k \cap \Lambda_P}(\pi_{D-k}(\mathbf{y}, P)). \tag{3.1}$$

3.1. On lattices

We know that Λ_k is a lattice of V_k and Λ^k is a lattice of V^k . In this subsection, we develop a better understanding of Λ_P and $\Lambda^k \cap \Lambda_P$.

We say a subspace $U \subset V$ is *rational* if it is spanned by a set of vectors with rational coordinates. It is easy to check that a subspace is rational if and only if it is defined by a set of linear equations with rational coefficients.

We state the following three fundamental lemmas about rational subspaces and lattices without proofs.

Lemma 3.3. *A subspace U of V is rational if and only if there exists a sublattice Γ of Λ such that Γ is a lattice of U , or equivalently $\Lambda_U := U \cap \Lambda$ is a lattice of U .*

Lemma 3.4. Suppose Γ_i is a sublattice of Λ , and U_i is the subspace of V spanned by Γ_i , for $i = 1, 2$. Then $\Gamma_1 \cap \Gamma_2$ is a lattice of $U_1 \cap U_2$.

Lemma 3.5. Suppose U is a rational subspace of V and $\Lambda_U = U \cap \Lambda$. (Note that Λ_U is a lattice of U .) Then any basis of Λ_U can be extended to a basis of Λ .

The next lemma discusses the lattice of the projection of a subspace and the lattice of the kernel of the projection.

Lemma 3.6. Suppose Γ is a sublattice of Λ and U is the subspace of V spanned by Γ . Then $\pi^{(D-k)}(\Gamma)$ is a lattice of $\pi^{(D-k)}(U)$ and $\Lambda^k \cap \Gamma$ is a lattice of $V^k \cap U$.

Let $r = \dim(\pi^{(D-k)}(U))$ and $t = \dim(V^k \cap U)$. Then $r + t = \dim(U)$.

Furthermore, suppose $(\mathbf{f}^1, \dots, \mathbf{f}^r)$ is a basis of the lattice $\Lambda^k \cap \Gamma$ and $(\mathbf{g}^1, \dots, \mathbf{g}^r)$ is a set of points in Γ . Then $(\mathbf{g}^1, \dots, \mathbf{g}^r, \mathbf{f}^1, \dots, \mathbf{f}^r)$ is a basis of Γ if and only if $(\pi^{(D-k)}(\mathbf{g}^1), \pi^{(D-k)}(\mathbf{g}^2), \dots, \pi^{(D-k)}(\mathbf{g}^r))$ is a basis of the lattice $\pi^{(D-k)}(\Gamma)$.

Proof. Space $V^k \cap U$ is the kernel of the projection $\pi^{(D-k)} : U \rightarrow \pi^{(D-k)}(U)$. Thus, we can decompose the lattice Γ as $\pi^{(D-k)}(\Gamma) \oplus (\Lambda^k \cap \Gamma)$. \square

Lemma 3.7. Suppose P is a d -dimensional central rational polytope in V . Then

- (i) $\Lambda_P = \text{aff}(P) \cap \Lambda$ is a lattice of $\text{aff}(P) = \text{lin}(P)$.
- (ii) $\pi^{(D-k)}(\Lambda_P)$ is a lattice of $\pi^{(D-k)}(\text{aff}(P))$.
- (iii) $\Lambda^k \cap \Lambda_P$ is a lattice of $V^k \cap \text{aff}(P)$.
- (iv) $\dim(\pi^{(D-k)}(\text{aff}(P))) + \dim(V^k \cap \text{aff}(P)) = d$.

Proof. Since P is central and rational, we have that $\text{aff}(P) = \text{lin}(P)$ is rational. Thus, (i) follows from Lemma 3.3, and (ii), (iii) and (iv) follow from Lemma 3.6. \square

3.2. On integrality and general position

We say an ℓ -dimensional subspace U of V is the *row space* of an $\ell \times D$ matrix if the rows of the matrix are the coordinates of a basis of U with respect to ϵ .

Recall a *unimodular matrix* is a square integer matrix with determinant $+1$ or -1 .

The following two lemmas can be checked directly from the definition.

Lemma 3.8. Let U be an ℓ -dimensional subspace of V . Then the following are equivalent:

- (i) U is integral.
- (ii) U is the row space of a matrix of the form $(I \ J)$, where I is the $\ell \times \ell$ identity matrix and J is an $\ell \times (D - \ell)$ integer matrix.
- (iii) U is the row space of a matrix of the form $(K \ J)$, where K is an $\ell \times \ell$ unimodular matrix and J is an $\ell \times (D - \ell)$ matrix of integer entries.

An affine space W is integral if and only if W contains a lattice point and $\text{lin}(W)$ is integral if and only if $\text{lin}(W)$ is integral and $W = \text{lin}(W) + \mathbf{z}$ for some (equivalently all) $\mathbf{z} \in W \cap \Lambda$.

Lemma 3.9. *Let U be an ℓ -dimensional subspace of V . Then the following are equivalent:*

- (i) U is in general position.
- (ii) U is the row space of a matrix of the form $(I \ J)$, where I is the $\ell \times \ell$ identity matrix and J is an arbitrary $\ell \times (D - \ell)$ matrix.
- (iii) U is the row space of a matrix of the form $(K \ J)$, where K is an invertible matrix and J is an arbitrary $\ell \times (D - \ell)$ matrix.
- (iv) If U is the row space of a matrix of the form $(K \ J)$, where J is any arbitrary $\ell \times (D - \ell)$ matrix, then K is an invertible matrix.

An affine space W is in general position if and only if $\text{lin}(W)$ is in general position.

Corollary 3.10. *Suppose $0 \leq \ell \leq m \leq D$ and W is an ℓ -dimensional affine space of V .*

- (i) *If W is integral, then $\text{lin}(W)$ is rational and W is in general position.*
- (ii) *If W is integral, then the projection $\pi^{(D-m)}(W)$ is ℓ -dimensional and integral. In particular, $\pi^{(D-\ell)}$ induces a bijection between $W \cap \Lambda$ and Λ_ℓ .*
- (iii) *If W is in general position, then the projection $\pi^{(D-m)}(W)$ is ℓ -dimensional and in general position. In particular, $\pi^{(D-\ell)}$ induces a bijection between W and V_ℓ .*
- (iv) *If W is in general position, for any $\ell + 1$ affinely independent points $\mathbf{x}_1, \dots, \mathbf{x}_{\ell+1} \in W$, we have that the matrix*

$$\begin{pmatrix} 1 & \pi^{(D-\ell)}(\mathbf{x}_1) \\ 1 & \pi^{(D-\ell)}(\mathbf{x}_2) \\ \vdots & \vdots \\ 1 & \pi^{(D-\ell)}(\mathbf{x}_{\ell+1}) \end{pmatrix}$$

is invertible.

Proof. (i), (ii) and (iii) directly follow from Lemma 3.8 and Lemma 3.9. One checks that

$$\det \begin{pmatrix} 1 & \pi^{(D-\ell)}(\mathbf{x}_1) \\ 1 & \pi^{(D-\ell)}(\mathbf{x}_2) \\ \vdots & \vdots \\ 1 & \pi^{(D-\ell)}(\mathbf{x}_{\ell+1}) \end{pmatrix} = (-1)^\ell \det \begin{pmatrix} \pi^{(D-\ell)}(\mathbf{x}_1) - \pi^{(D-\ell)}(\mathbf{x}_{\ell+1}) \\ \pi^{(D-\ell)}(\mathbf{x}_2) - \pi^{(D-\ell)}(\mathbf{x}_{\ell+1}) \\ \vdots \\ \pi^{(D-\ell)}(\mathbf{x}_\ell) - \pi^{(D-\ell)}(\mathbf{x}_{\ell+1}) \end{pmatrix},$$

and $\text{lin}(W)$ is the row space of the matrix $(\mathbf{x}_i - \mathbf{x}_{\ell+1})_{1 \leq i \leq \ell}$, whose first ℓ columns are exactly the latter matrix in the above formula. Then (iv) follows from Lemma 3.9(iv). \square

Given (i) of Corollary 3.10, we can apply all the results we obtain for affine spaces (or polytopes) in general position to affine spaces (or polytopes) that are integral. This is an important fact which we use implicitly throughout the paper.

Lemma 3.11. *Suppose $P \subset V$ is a d -dimensional rational polytope.*

- (i) *If one of the k -faces of P is affinely integral, then $\pi^{(D-k)}(\Lambda_P) = \Lambda_k$.*

(ii) If P is central and one of the k -faces of P is in affinely general position, then $\pi^{(D-k)}(\Lambda_P)$ is of rank k and $\Lambda^k \cap \Lambda_P$ is of rank $d - k$.

Proof. (i) follows immediately from Corollary 3.10(ii).
 (ii) follows from Corollary 3.10(iii) and Lemma 3.7. \square

Lemma 3.12. Suppose P is a k -integral polytope. Then mP is a k -integral polytope, for any positive integer m .

Proof. This can be checked using Lemma 3.8. \square

3.3. Algebra of polyhedra

Both the number of lattice points and the volume of a polytope are *valuations*; that is, they satisfy the inclusion–exclusion property. Since the theory of valuations is based on the theory of the algebra of polyhedra, we will recall the basic definitions in the area. Please refer to [4] for details.

Definition 3.13. Let $A \subset V$ be a set. The *indicator function*, or just *indicator*, of A is the function $[A] : V \rightarrow \mathbb{R}$ defined by

$$[A](x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

The *algebra of polyhedra* $\mathcal{P}(V)$ is the vector space over \mathbb{Q} spanned by the indicator functions $[P]$ of all polyhedra $P \subset V$. The *algebra of polytopes* $\mathcal{P}_b(V)$ is the vector space over \mathbb{Q} spanned by the indicator functions $[P]$ of all polytopes $P \subset V$.

Definition 3.14. A linear transformation $\Phi : \mathcal{P}(V) \rightarrow V'$, or $\Phi : \mathcal{P}_b(V) \rightarrow V'$, where V' is a vector space, is called a *valuation*.

It is clear that one can extend the number of lattice points of a polytope to a valuation of $\mathcal{P}_b(V)$. However, we have to be more careful when we extend the volume function to a valuation, because if we use different measures for different polytopes, the linearity will fail. Thus, we fix a lattice in the following definition. For convenience, we also fix a notation related to the right side of (3.1).

Definition 3.15. Let Γ be a sublattice of Λ and U the subspace of V spanned by Γ . For any polytope $P \subset U$, we define the *k th S -volume of P (with respect to Γ)* to be

$$S\text{Vol}_\Gamma^k(P) = \sum_{\mathbf{y} \in \pi^{(D-k)}(\Gamma)} \text{Vol}_{\Lambda^k \cap \Gamma}(\pi_{D-k}(\mathbf{y}, P)).$$

Recall $\text{Vol}_\Gamma(P)$ is defined by (1.1). One can extend both Vol_Γ and $S\text{Vol}_\Gamma^k$ to valuations of $\mathcal{P}_b(U)$.

For convenience, for any $[S] \in \mathcal{P}_b(U)$, we omit the bracket and write $\text{Vol}_\Gamma(S)$ and $S\text{Vol}_\Gamma^k(S)$ instead of $\text{Vol}_\Gamma([S])$ and $S\text{Vol}_\Gamma^k([S])$.

Using the definition of SVol_Γ^k and Lemma 3.2, we can rephrase Theorem 1.5 as the following proposition.

Proposition 3.16. *Suppose $0 < k < d \leq D$ and $P \subset V$ is a d -dimensional central $(k - 1)$ -integral polytope in k -general position. Let $\Lambda_P := \text{aff}(P) \cap \Lambda$. Then the volume of P is given by*

$$\text{Vol}_{\Lambda_P}(P) = \text{SVol}_{\Lambda_P}^k(P). \tag{3.2}$$

In the next three sections, we will prove a special case of Proposition 3.16: when $k = 1$. The material in these sections is not used in the subsequent sections, so the reader who is willing to take this case of Proposition 3.16 on faith may skip to Section 7.

All of the results in Sections 4 and 5 and some of the results in Section 6 are stated for general k , since the statements and proofs are no harder than the $k = 1$ case.

4. Affine transformations that preserve certain properties

In this section, we will discuss conditions for affine transformations that preserve certain properties of polytopes: integrality, general position, volumes and S-volumes. This serves as a preparation for the reduction we will discuss in the next section.

Suppose Γ_i is a sublattice of Λ which has a basis $\mathbf{e}_i = (\mathbf{e}_i^1, \dots, \mathbf{e}_i^{D_i})$, and U_i the subspace of V spanned by Γ_i , for $i = 1, 2$. For any affine transformation $\phi : U_1 \rightarrow U_2$, we can associate a vector $\alpha_{\mathbf{e}_1, \mathbf{e}_2}^\phi \in \mathbb{R}^{D_2}$ and a $D_1 \times D_2$ matrix $M_{\mathbf{e}_1, \mathbf{e}_2}^\phi$ to ϕ such that ϕ sends a point in U_1 with coordinates \mathbf{x} with respect to the basis \mathbf{e}_1 to the point in U_2 with coordinates $\alpha_{\mathbf{e}_1, \mathbf{e}_2}^\phi + \mathbf{x}M_{\mathbf{e}_1, \mathbf{e}_2}^\phi$ with respect to the basis \mathbf{e}_2 . One verifies that

$$M_{\mathbf{e}_1, \mathbf{e}_2}^\phi = (m_{i,j}), \quad \text{if } \phi(\mathbf{e}_1^i) - \phi(0) = \sum_{j=1}^{D_2} m_{i,j} \mathbf{e}_2^j,$$

and $\alpha_{\mathbf{e}_1, \mathbf{e}_2}^\phi$ are the coordinates of $\phi(0)$ with respect to \mathbf{e}_2 .

If the map is from V to V , unless otherwise noted we will assume that the lattice is Λ with the fixed basis $\mathbf{e} = (\mathbf{e}^1, \dots, \mathbf{e}^D)$ and omit the subscript “ $\mathbf{e}_1, \mathbf{e}_2$ ”. Similarly, we can associate a matrix to any linear transformation.

4.1. Conditions for preserving integrality and general position

We say an affine map ϕ *preserves integrality* of an integral affine space W if $\phi(W)$ is integral. Similarly, we say an affine map ϕ *preserves generality* of an affine space W in general position, if $\phi(W)$ is in general position.

Lemma 4.1. *Suppose $\phi : V \rightarrow V$ is an affine map with associated matrix*

$$M = \begin{pmatrix} A_{\ell \times \ell} & C_{\ell \times (D-\ell)} \\ O_{(D-\ell) \times \ell} & B_{(D-\ell) \times (D-\ell)} \end{pmatrix}$$

and vector α , where O is the zero matrix and B is invertible.

- (i) Let W be an ℓ -dimensional integral affine space. If A is unimodular and ϕ sends any lattice point in W to a lattice point, then ϕ preserves integrality of W .
- (ii) ϕ preserves integrality of any ℓ -dimensional integral affine spaces of V if A is unimodular, and both B and C are integer matrices, and α is an integer vector.
- (iii) ϕ preserves generality of any ℓ -dimensional affine spaces of V in general position if A is invertible.

Proof. (ii) follows from (i), and (iii) can be check directly using Lemma 3.9(iii). Now we prove (i). Let $U = \text{lin}(W)$, then by Lemma 3.8, U is integral and $W = U + \mathbf{z}$, for some $\mathbf{z} \in W \cap \Lambda$. By Lemma 3.8(ii), U is the row space of an integer matrix of the form $(I \ J)$, where I is the $\ell \times \ell$ identity matrix. One checks that $\phi(W) = U' + \phi(\mathbf{z})$, where U' is the row space of $(I \ J)M = (A \ C + JB)$ and $\phi(\mathbf{z}) \in \phi(W) \cap \Lambda$. Thus, it is enough to show that U' is integral. Since A is unimodular, by Lemma 3.8(iii), we only need to verify that $C + JB$ is an integer matrix, or equivalently each row in $(I \ J)M$ is an integer vector. Thus, we need to show that for any row vector \mathbf{v} in $(I \ J)$, $\mathbf{v}M$ is an integer vector. However, \mathbf{v} is an integer vector and W contains lattice points, so \mathbf{v} can be written as the difference of two integer vectors \mathbf{u}_1 and \mathbf{u}_2 , each of which is the coordinates of a lattice point in W . Because ϕ sends lattice points in W to lattice points, we have that $\alpha + \mathbf{u}_i M$ is an integer vector, for each i . Therefore, $\mathbf{v}M$ is an integer vector as well. \square

The following corollary follows immediately.

Corollary 4.2. Suppose $\phi : V \rightarrow V$ is an invertible affine map with an associated matrix

$$M = \begin{pmatrix} A_{k \times k} & C_{k \times (D-k)} \\ O_{(D-k) \times k} & B_{(D-k) \times (D-k)} \end{pmatrix}$$

and vector α , where O is the zero matrix. (We must have that A and B are invertible.)

- (i) Suppose W is an affine space of V satisfying that ϕ sends any lattice point in W to a lattice point. If A is an upper triangular and unimodular matrix, then ϕ preserves integrality of any integral affine space of W of dimension no more than k .
- (ii) If A is an upper triangular and unimodular matrix, both B and C are integer matrices, and α is an integer vector, then ϕ preserves integrality of any integral affine space of V of dimension no more than k .
- (iii) If A is upper triangular, then ϕ preserves generality of any affine space of V of dimension no more than k that is in general position.

4.2. Conditions for preserving volumes and S-volumes

We will discuss situations when $\text{Vol}_\Gamma(P)$ and $\text{SVol}_\Gamma^k(P)$ are invariant under affine transformation.

Lemma 4.3. Let s be a positive integer. For $i = 1, 2$, let Γ_i be a sublattice of Λ of rank s and U_i the subspace of V spanned by Γ_i . Suppose $\mathbf{e}_i = (\mathbf{e}_i^1, \dots, \mathbf{e}_i^s)$ is a basis of Γ_i , for $i = 1, 2$. Let $\phi : U_1 \rightarrow U_2$ be an invertible affine map with the associated matrix $M_{\mathbf{e}_1, \mathbf{e}_2}$.

- (i) If $\det(M_{\mathbf{e}_1, \mathbf{e}_2}) = 1$, then $\text{Vol}_{\Gamma_1} = \text{Vol}_{\Gamma_2} \circ \phi$.
- (ii) Suppose the following four conditions are satisfied:
 - (1) $\pi^{(D-k)}(\mathbf{x}_1) = \pi^{(D-k)}(\mathbf{x}_2)$ if and only if $\pi^{(D-k)}(\phi(\mathbf{x}_1)) = \pi^{(D-k)}(\phi(\mathbf{x}_2))$.
 - (2) $\pi^{(D-k)}(\mathbf{x}) \in \pi^{(D-k)}(\Gamma_1)$ if and only if $\pi^{(D-k)}(\phi(\mathbf{x})) \in \pi^{(D-k)}(\Gamma_2)$.
 - (3) For some number r and for $i = 1, 2$, we have that $(\mathbf{e}_i^{r+1}, \mathbf{e}_i^{r+2}, \dots, \mathbf{e}_i^s)$ is a basis of the lattice $\Lambda^k \cap \Gamma_i$. (Thus, it spans $V^k \cap U_i$ by Lemma 3.4.)
 - (4) The determinant of the lower right $(s - r) \times (s - r)$ submatrix of $M_{\mathbf{e}_1, \mathbf{e}_2}$ is equal to 1. Then $\text{SVol}_{\Gamma_1}^k = \text{SVol}_{\Gamma_2}^k \circ \phi$.

Proof. (i) follows directly from the definition of volume and linear algebra.

By (1), one sees that ϕ induces a bijection between $\pi^{(D-k)}(U_1)$ and $\pi^{(D-k)}(U_2)$. We denote this map by ϕ' . We then have that for any set $S \in U_1$ and $\mathbf{y} \in \pi^{(D-k)}(U_1)$,

$$\pi_{D-k}(\phi'(\mathbf{y}), \phi(S)) = \phi(\pi_{D-k}(\mathbf{y}, S)).$$

Note that shifting a set of points by a constant vector does not change its volume. Thus,

$$\text{Vol}_{\Lambda^k \cap \Gamma_2}(\phi(\pi_{D-k}(\mathbf{y}, S))) = \text{Vol}_{\Lambda^k \cap \Gamma_2}(\{\mathbf{x} - \phi'(\mathbf{y}) \mid \mathbf{x} \in \phi(\pi_{D-k}(\mathbf{y}, S))\}), \tag{4.1}$$

and

$$\text{Vol}_{\Lambda^k \cap \Gamma_1}(\pi_{D-k}(\mathbf{y}, S)) = \text{Vol}_{\Lambda^k \cap \Gamma_1}(\{\mathbf{x} - \mathbf{y} \mid \mathbf{x} \in \pi_{D-k}(\mathbf{y}, S)\}). \tag{4.2}$$

We will show the right sides of the above two equations are equal. We define an affine map $\phi_{\mathbf{y}} : V^k \cap U_1 \rightarrow V^k \cap U_2$ by mapping \mathbf{z} to $\phi(\mathbf{y} + \mathbf{z}) - \phi'(\mathbf{y})$. Using the bases $(\mathbf{e}_1^{r+1}, \mathbf{e}_1^{r+2}, \dots, \mathbf{e}_1^s)$ and $(\mathbf{e}_2^{r+1}, \mathbf{e}_2^{r+2}, \dots, \mathbf{e}_2^s)$, one can check the matrix associated to $\phi_{\mathbf{y}}$ is exactly the lower right $(s - r) \times (s - r)$ submatrix of $M_{\mathbf{e}_1, \mathbf{e}_2}$. Hence, by (4), we have that

$$\text{Vol}_{\Lambda^k \cap \Gamma_1} = \text{Vol}_{\Lambda^k \cap \Gamma_2} \circ \phi_{\mathbf{y}}.$$

We verify that

$$\phi_{\mathbf{y}}(\{\mathbf{x} - \mathbf{y} \mid \mathbf{x} \in \pi_{D-k}(\mathbf{y}, S)\}) = \{\mathbf{x} - \phi'(\mathbf{y}) \mid \mathbf{x} \in \phi(\pi_{D-k}(\mathbf{y}, S))\}.$$

Therefore, the right sides of Eqs. (4.1) and (4.2) are equal. So the left sides are equal, and we conclude that for any set $S \in U_1$ and $\mathbf{y} \in \pi^{(D-k)}(U_1)$,

$$\begin{aligned} \text{Vol}_{\Lambda^k \cap \Gamma_2}(\pi_{D-k}(\phi'(\mathbf{y}), \phi(S))) &= \text{Vol}_{\Lambda^k \cap \Gamma_2}(\phi(\pi_{D-k}(\mathbf{y}, S))) \\ &= \text{Vol}_{\Lambda^k \cap \Gamma_1}(\pi_{D-k}(\mathbf{y}, S)). \end{aligned}$$

Furthermore, by (2), ϕ' also gives a bijection between $\pi^{(D-k)}(\Gamma_1)$ and $\pi^{(D-k)}(\Gamma_2)$.

Thus, we have $\text{SVol}_{\Gamma_1}^k = \text{SVol}_{\Gamma_2}^k \circ \phi$. \square

Corollary 4.4. *Let s be a positive integer. For $i = 1, 2$, let Γ_i be a sublattice of Λ of rank s and U_i the subspace of V spanned by Γ_i . Suppose $\mathbf{e}_i = (\mathbf{e}_i^1, \dots, \mathbf{e}_i^s)$ is a basis of Γ_i , and for some number r we have that $(\mathbf{e}_i^{r+1}, \mathbf{e}_i^{r+2}, \dots, \mathbf{e}_i^s)$ is a basis of the lattice $\Lambda^k \cap \Gamma_i$. Suppose $\phi : U_1 \rightarrow U_2$ is an affine map with an associated matrix*

$$M_{\mathbf{e}_1, \mathbf{e}_2}^\phi = \begin{pmatrix} A & C \\ O & B \end{pmatrix}$$

and vector $\alpha_{\mathbf{e}_1, \mathbf{e}_2} = (\alpha_1, \alpha_2)$ satisfying

- (i) A is an $r \times r$ unimodular matrix.
- (ii) α_1 is an r -dimensional vector of integer entries.
- (iii) B is an $(s - r) \times (s - r)$ matrix with $\det(B) = 1$.
- (iv) O is the zero matrix.

Then $\text{Vol}_{\Gamma_1} = \text{Vol}_{\Gamma_2} \circ \phi$ and $\text{SVol}_{\Gamma_1}^k = \text{SVol}_{\Gamma_2}^k \circ \phi$.

Proof. It is clear that $\det(M_{\mathbf{e}_1, \mathbf{e}_2}) = 1$. Thus, $\text{Vol}_{\Gamma_1} = \text{Vol}_{\Gamma_2} \circ \phi$. To show that $\text{SVol}_{\Gamma_1}^k = \text{SVol}_{\Gamma_2}^k \circ \phi$, we need to verify (1)–(4) under (ii) of Lemma 4.3 holds. (3) and (4) are given in the conditions.

By Lemma 3.6, $\mathbf{f}_i = (\pi^{(D-k)}(\mathbf{e}_i^1), \dots, \pi^{(D-k)}(\mathbf{e}_i^r))$ is a basis of $\pi^{(D-k)}(\Gamma_i)$, for $i = 1, 2$. One checks that ϕ deduces an affine map from $\pi^{(D-k)}(U_1)$ to $\pi^{(D-k)}(U_2)$ by mapping $\pi^{(D-k)}(\mathbf{x})$ to $\pi^{(D-k)}(\phi(\mathbf{x}))$. The matrix and vector associated to this map with respect to the bases \mathbf{f}_1 and \mathbf{f}_2 are A and α_1 , respectively. Therefore, (1) and (2) of Lemma 4.3 are satisfied. \square

5. Reduction to the full-dimensional case

In this section, we will prove the following proposition which reduces Proposition 3.16 to the case of full-dimensional polytopes.

Proposition 5.1. *Suppose $0 < k < d \leq D$ and $P \subset V$ is a d -dimensional central $(k - 1)$ -integral polytope in k -general position. There exists an invertible affine map $\phi : V \rightarrow V$ such that $Q := \phi(P)$ is a full-dimensional polytope in V_d that is still $(k - 1)$ -integral and in k -general position. Also, proving Proposition 3.16 for P is equivalent to proving it for Q .*

Throughout this section, we will assume that $0 < k < d \leq D$ and $P \subset V$ is a d -dimensional central $(k - 1)$ -integral polytope in k -general position. We also fix

$$U_P := \text{aff}(P),$$

which is a subspace of V because P is central.

We first prove two preliminary lemmas.

Lemma 5.2. *There exist integers h_1, \dots, h_k such that*

$$(\mathbf{e}^1 + h_1 \mathbf{e}^k, \dots, \mathbf{e}^{k-1} + h_{k-1} \mathbf{e}^k, h_k \mathbf{e}^k) \text{ is a basis of } \pi^{(D-k)}(\Lambda_P).$$

Proof. By Lemma 3.11, we have that $\pi^{(D-(k-1))}(\Lambda_P) = \Lambda_{k-1}$, and $\pi^{(D-k)}(\Lambda_P)$ is of rank k . An obvious basis for $\pi^{(D-(k-1))}(\Lambda_P)$ is $(\mathbf{e}^1, \mathbf{e}^2, \dots, \mathbf{e}^{k-1})$. Note that $\pi^{(D-(k-1))}(\Lambda_P)$ can be obtained by dropping the last coordinate, i.e., the coordinate corresponding to \mathbf{e}^k , of the lattice $\pi^{(D-k)}(\Lambda_P)$. Hence, our lemma follows from Lemma 3.6. \square

Lemma 5.3. *There exists a basis $\mathfrak{f} = (\mathbf{f}^1, \mathbf{f}^2, \dots, \mathbf{f}^D)$ of V satisfying*

- (i) $\pi^{(D-k)}(\mathbf{f}^i) = \mathbf{e}^i + h_i \mathbf{e}^k$, for $i = 1, \dots, k - 1$, and $\pi^{(D-k)}(\mathbf{f}^k) = h_k \mathbf{e}^k$, for some integers h_1, \dots, h_k .
- (ii) $(\mathbf{f}^{k+1}, \dots, \mathbf{f}^d)$ is a basis of the lattice $\Lambda^k \cap \Lambda_P$.
- (iii) $(\mathbf{f}^1, \dots, \mathbf{f}^k, \mathbf{f}^{k+1}, \dots, \mathbf{f}^d)$ is a basis of the lattice Λ_P .
- (iv) $(\mathbf{f}^{k+1}, \dots, \mathbf{f}^d, \mathbf{f}^{d+1}, \dots, \mathbf{f}^D)$ is a basis of the lattice Λ^k .

Proof. By Lemma 5.2, $\pi^{(D-k)}(\Lambda_P)$ has a basis

$$(\mathbf{e}_1 + h_1 \mathbf{e}^k, \dots, \mathbf{e}_{k-1} + h_{k-1} \mathbf{e}^{k-1}, h_k \mathbf{e}^k),$$

for some integers h_1, \dots, h_k . Thus, using Lemma 3.6 again, we can choose $\mathbf{f}^1, \dots, \mathbf{f}^k, \mathbf{f}^{k+1}, \dots, \mathbf{f}^d$ satisfying (i), (ii) and (iii).

Furthermore, by Lemma 3.5, $(\mathbf{f}^{k+1}, \dots, \mathbf{f}^d)$ can be extended to a basis, say $(\mathbf{f}^{k+1}, \dots, \mathbf{f}^d, \mathbf{f}^{d+1}, \dots, \mathbf{f}^D)$ of Λ^k . \square

Proof of Proposition 5.1. Let $\mathfrak{f} = (\mathbf{f}^1, \dots, \mathbf{f}^D)$ be a basis as described in Lemma 5.3, and define $\phi : V \rightarrow V$ to be the linear transformation that maps \mathbf{f}^i to \mathbf{e}^i for each i . Both \mathfrak{f} and \mathbf{e} are bases of V . Thus, ϕ is invertible. Let $Q = \phi(P)$. Clearly, Q is a d -dimensional polytope in V_d .

One checks that the associated matrix to ϕ is $M^\phi = M_{\mathbf{e}, \mathbf{f}}^\phi = F^{-1}$, where F is the $D \times D$ matrix whose i th row is the coordinates of \mathbf{f}^i with respect to \mathbf{e} . By Lemma 5.3, we know that F is a matrix satisfying the following three properties:

- (i) The lower left $(D - k) \times k$ submatrix of F is a zero matrix.
- (ii) The upper left $k \times k$ submatrix of F is an upper triangular matrix.
- (iii) The upper left $(k - 1) \times (k - 1)$ submatrix of F is an upper triangular unimodular matrix.

It is easy to verify that $M^\phi = F^{-1}$ also has these three properties. Recall that $U_P = \text{aff}(P)$ is a subspace of V , and $\Lambda_P = \text{aff}(P) \cap \Lambda$. Hence, by the construction of ϕ , any lattice point in U_P is mapped to a lattice point by ϕ . Therefore, by using (i) and (iii) of Corollary 4.2, we conclude that Q is a $(k - 1)$ -integral polytope in k -general position.

The map ϕ induces a linear map $\tilde{\phi} : U_P \rightarrow V_d$. It is clear that the associated matrix $M_{\tilde{\mathfrak{f}}, \tilde{\mathbf{e}}}^{\tilde{\phi}}$ to $\tilde{\phi}$ with respect to the bases $\tilde{\mathfrak{f}} = (\mathbf{f}^1, \dots, \mathbf{f}^d)$ and $\tilde{\mathbf{e}} = (\mathbf{e}^1, \dots, \mathbf{e}^d)$ is the identity matrix. Note that by (ii) of Lemma 5.3, $(\mathbf{f}^{k+1}, \dots, \mathbf{f}^d)$ is a basis of $\Lambda^k \cap \Lambda_P$, and it is obvious that $(\mathbf{e}^{k+1}, \dots, \mathbf{e}^d)$ is a basis of $\Lambda^k \cap \Lambda_d$. Thus, by Corollary 4.4, $\text{Vol}_{\Lambda_P}(P) = \text{Vol}_{\Lambda_d}(Q)$ and $\text{SVol}_{\Lambda_P}^k(P) = \text{SVol}_{\Lambda_d}^k(Q)$. However, Q is full-dimensional in V_d . Thus, $\Lambda_Q = V_d \cap \Lambda = \Lambda_d$.

Thus, we conclude that proving Proposition 3.16 for P is reduced to proving it for $Q = \phi(P)$. Therefore, Proposition 5.1 follows the fact that Q is full-dimensional in V_d . \square

6. A special case of Proposition 3.16

In this section, we prove the following proposition as a special case of Proposition 3.16.

Proposition 6.1. *Proposition 3.16 is true when $k = 1$.*

We will give two proofs of Proposition 6.1 in two subsections: the first one was suggested by the referee, and the second one is our original proof which we will only present the idea.

6.1. First proof

We use Lemma 2.3 of [2] by Barvinok. Since the notation in [2] is different from ours and we only need a special case of the lemma, we present a special case of the lemma in our notation:

Lemma 6.2. *(See Lemma 2.3 of [2].) Let $P \subset V$ be a full-dimensional rational polytope and let $Q = \pi^{(d-k)}(P)$. Suppose $\partial Q \cap \Lambda_k = \emptyset$. Then*

$$\text{SVol}_\Lambda^k(P) = \lim_{\tau \rightarrow +\infty} \sum_{l \in \Lambda_k} \exp\{-\pi \|l\|^2/\tau\} \int_P \exp\{2\pi i \langle l, x \rangle\} d\Lambda, \tag{6.1}$$

where $\langle l, x \rangle$ is the dot product of the coordinates of l and x .

Remark 6.3. After the proof of Lemma 2.3 in [2], Barvinok also remarked that if $\partial Q \cap \Lambda_k \neq \emptyset$, the right-hand side of Eq. (6.1) still converges and the limit is equal to a modified version of SVol_Λ^k , with appropriate weights on the volumes of slices of P over points in $\partial Q \cap \Lambda_k$. Therefore, we conclude that if the polytope P satisfies the condition that the dimension of $\pi_{d-k}(\mathbf{y}, P)$ is less than $d - k$ for any lattice point \mathbf{y} in the boundary of $Q = \pi^{(d-k)}(P)$, then (6.1) still holds.

We also need the following lemma in order to prove Proposition 6.1.

Lemma 6.4. *Let $P \subset V$ be an integral d -dimensional polytope and $l \in V$. Suppose $\langle l, x \rangle$ is not constant on any edge of P , and $\langle l, x \rangle$ is an integer for any $x \in \Lambda_P$. Then*

$$\int_P \exp\{2\pi i \langle l, x \rangle\} d\Lambda_{\text{lin}(P)} = \delta_{0,d}. \tag{6.2}$$

Proof. It is clear that if P is 0-dimensional, then (6.2) holds. For $d \geq 1$, one sees that it is enough to prove the case when P is central because shifting P by a lattice point won't change the value of $\int_P \exp\{2\pi i \langle l, x \rangle\} d\Lambda_{\text{lin}(P)}$. We will prove (6.2) by induction on d using another result by Barvinok obtained from Stokes formula [1, Lemma 2.5]:

$$\int_P \exp\{2\pi i \langle l, x \rangle\} d\Lambda_{\text{lin}(P)} = \frac{1}{2\pi i \langle l, \lambda \rangle} \sum_{F: \text{facet of } P} \langle \lambda, n_F \rangle \int_F \exp\{2\pi i \langle l, x \rangle\} d\Lambda_{\text{lin}(F)}, \tag{6.3}$$

where $l, \lambda \in \text{lin}(P)$ are any vectors satisfying $\langle l, \lambda \rangle \neq 0$, and n_F is the outer unit normal to F .

Note that we can always assume $l \in \text{lin}(P)$: For any $l \in V$, one can let l' be the orthogonal projection of l onto $\text{lin}(P)$. Then $\langle l, x \rangle = \langle l', x \rangle$ for any $x \in \text{lin}(P) = \text{aff}(P)$.

Suppose $d = 1$. Then P is an interval with two vertices v_1 and v_2 , for some $v_1, v_2 \in \Lambda$. Clearly $n_{v_1} = -n_{v_2}$. Since $\langle l, x \rangle$ is not constant on $P = \text{conv}(v_1, v_2)$, we have that $\langle l, n_{v_1} \rangle \neq 0$. Hence, we can choose $\lambda = n_{v_1}$. Thus,

$$\int_P \exp\{2\pi i \langle l, x \rangle\} d\Lambda_{\text{lin}(P)} = \frac{1}{2\pi i \langle l, \lambda \rangle} (\langle \lambda, n_{v_1} \rangle + \langle \lambda, n_{v_2} \rangle) = 0.$$

Now we assume (6.2) holds for P of dimension less than d . We can choose λ to be the direction of an edge of P . Then (6.2) follows immediately from the induction hypothesis and (6.3). \square

Proof of Proposition 6.1. By Proposition 5.1, we can assume that $d = D$, i.e., P is a full-dimensional polytope. Hence, $\Lambda_P = \Lambda$.

Let $Q = \pi^{(d-1)}(P)$. Because $\pi^{(d-1)}$ is an open map, for any $\mathbf{y} \in \partial Q$, the slice $\pi_{d-1}(\mathbf{y}, P)$ only consists of boundary points of P . Because P is in 1-general position, $\pi_{d-1}(\mathbf{y}, P)$ cannot contain more than one point. Hence, the dimension of $\pi_{d-1}(\mathbf{y}, P)$ is 0, which is less than $d - 1$ since $1 < d$. Therefore, by Lemma 6.2 and Remark 6.3, we have

$$\text{SVol}_\Lambda^1(P) = \lim_{\tau \rightarrow +\infty} \sum_{l \in \Lambda_1} \exp\{-\pi \|l\|^2/\tau\} \int_P \exp\{2\pi i \langle l, x \rangle\} d\Lambda. \tag{6.4}$$

Let $l \in \Lambda_1$. If $l = 0$, then $\int_P \exp\{2\pi i \langle l, x \rangle\} d\Lambda = \text{Vol}_\Lambda(P)$. If $l \neq 0$, then because P is in 1-general position, l satisfies the hypothesis in Lemma 6.4. Therefore, $\int_P \exp\{2\pi i \langle l, x \rangle\} d\Lambda = 0$ for $l \neq 0$. Hence,

$$\text{SVol}_\Lambda^1(P) = \lim_{\tau \rightarrow +\infty} \exp\{-\pi \|0\|^2/\tau\} \text{Vol}_\Lambda(P) = \text{Vol}_\Lambda(P). \quad \square$$

6.2. Second proof

In this subsection, we present the idea of our original proof of Proposition 6.1. Since we have already given a complete proof in the last subsection, we omit the details. We want to present this proof because the proof uses very different techniques from the one in last subsection and although the full proof is somewhat longer, the idea is completely elementary and does not require knowledge of the theory of valuations.

First, by discussing properties of being in k -general position, one can easily prove the following two lemmas.

Lemma 6.5. Suppose $0 < k < d$. Let P be a d -dimensional central rational polytope in V . Suppose P has a triangulation $\bigsqcup Q_i$ consisting of simplices in k -general position. Let $\Gamma = \Lambda_P$. It is clear that $\text{aff}(P) = \text{aff}(Q_i)$. So Q_i is central and $\Gamma = \Lambda_{Q_i}$. Then

$$\text{SVol}_\Gamma^k(P) = \sum \text{SVol}_\Gamma^k(Q_i).$$

Lemma 6.6. *Let P be a d -dimensional polytope in 1-general position. Then there exists a triangulation $\sqcup Q_i$ of P without introducing new vertices such that each Q_i is a simplex in 1-general position.*

Given Lemma 6.6 and Lemma 6.5 with $k = 1$, we can reduce Proposition 6.1 to the following proposition:

Proposition 6.7. *Proposition 3.16 is true when $k = 1$, and P is a simplex.*

Recall that Proposition 5.1 enables us to reduce the problem to full-dimensional polytopes. We can also show that the problem can be reduced further to polytopes in fully general position by using the corollaries in Section 4. Hence, we conclude that it is enough to prove Proposition 6.7 with the assumption that P is a full-dimensional simplex in fully general position.

Note that since $\Delta_P = \Delta$, what we need to show is that

$$\text{Vol}_\Delta(P) = \text{SVol}_\Delta^1(P). \tag{6.5}$$

However, since P is a full-dimensional simplex, the left-hand side of (6.5) is simply a determinant of a matrix whose entries are coordinates of the vertices of P .

The bulk of our proof consists of showing the right-hand side of (6.5) is given by the same determinant. First, by Theorem 4.6 in [9], we decompose P , a d -dimensional simplex in fully general position, into $d!$ signed sets, each of which corresponds to a permutation in the symmetric group S_d . Then we are able to write the right-hand side of (6.5) in terms of a summation involving signs of permutations, determinants and power sums. By analyzing this formula further and using Lemma 7.3 in [9], we are able to derive the desired result.

We conclude by remarking that with the above decomposition idea, one can prove that Proposition 3.16 is true for a simplex P without restricting to $k = 1$. However, unlike the case when $k = 1$, we do not see any obvious way to prove a k -version of Lemma 6.6 which will reduce the general case to the simplex case.

7. Properties of slices and projections

Before we move on to finishing the proof of Proposition 3.16 and thus Theorem 1.5, we will discuss properties of slices and projections of certain polytopes. The results presented in this section will be used both in the proof of Proposition 3.16 in Section 8 and in the proof of Theorem 1.3 in Section 9.

Let P be a polytope in V . One sees that the projection $\pi^{(D-m)}(P)$ is a polytope in V_m , and the slice $\pi_{D-m}(\mathbf{y}, P)$ is a polytope in $\pi_{D-m}(\mathbf{y}, V) \cong V^m$, for any $0 \leq m \leq D$ and $\mathbf{y} \in \pi^{(D-m)}(P)$. We will discuss properties of projections and slices of P under various conditions.

We denote by ∂P and $\text{Int}(P)$ the relative boundary and the relative interior of P . Since in this paper we only discuss the relative boundary and the relative interior of polytopes, we will omit “relative” and just say “boundary” and “interior”. We denote by $\mathcal{F}(P)$ and $\mathcal{F}_{\geq \ell}(P)$ the set of faces and the set of faces of dimension at least ℓ , respectively, of P .

Let $Y \subset V_m$, and $S \subset V$. We define

$$\pi_{D-m}(Y, S) = \bigcup_{\mathbf{y} \in Y} \pi_{D-m}(\mathbf{y}, S).$$

Lemma 7.1. *Suppose $0 < \ell \leq m \leq D$ and $P \subset V$ is a polytope whose ℓ -faces are all in affinely general position. For any $(\ell - 1)$ -face F_0 of $\pi^{(D-m)}(P)$, we have that $F := \pi_{D-m}(F_0, P)$ is an $(\ell - 1)$ -face of P .*

Therefore, $\pi_{D-m}(\mathbf{y}, P)$ is one point on the boundary of P , for any point \mathbf{y} of an $(\ell - 1)$ -face of $\pi^{(D-m)}(P)$.

Proof. Since F_0 is a face of $\pi^{(D-m)}(P)$, there exists an $(m - 1)$ -dimensional affine space H_0 in V_m such that $\pi^{(D-m)}(P)$ is on one side of H_0 and $F_0 = H_0 \cap \pi^{(D-m)}(P)$. Let H be the inverse image of H_0 under $\pi^{(D-m)}$. One sees that H is a $(D - 1)$ -dimensional affine space in V , of which P is on one side. It is clear that $F = \pi_{D-m}(F_0, P) = H \cap P$, thus is a face of P . Because $\pi^{(D-m)}(F) = F_0$, we must have that $\dim(F) \geq \dim(F_0) = \ell - 1$. Suppose $\dim(F) \geq \ell$. Let F' be an ℓ -face of F . Then F' is a face of P , thus is in affinely general position. Therefore, by Corollary 3.10(iii), $\dim(\pi^{(D-m)}(F')) = \ell$. However, $\pi^{(D-m)}(F') \subset \pi^{(D-m)}(F) = F_0$, which contradicts the hypothesis that F_0 has dimension $\ell - 1$. \square

Corollary 7.2. *Suppose $0 \leq k \leq d \leq D$ and $P \subset V$ is a k -integral polytope. Then $\pi^{(D-k)}(P)$ is a k -dimensional fully integral polytope.*

Proof. By Lemma 3.11(i), the projection $\pi^{(D-k)}(P)$ is k -dimensional. The only k -face of $\pi^{(D-k)}(P)$ is itself. Since $\text{aff}(\pi^{(D-k)}(P)) = V_k$, it is automatically integral. For any $\ell < k$, and any ℓ -face F_0 of $\pi^{(D-k)}(P)$, by Lemma 7.1, there exists an ℓ -face F of P , such that $F = \pi_{D-k}(F_0, P)$. Thus, $\pi^{(D-k)}(F) = F_0$. However, F is affinely integral, which means $\text{aff}(F)$ is integral. By Corollary 3.10(ii), $\pi^{(D-k)}(\text{aff}(F))$ is integral. One checks that $\text{aff}(F_0) = \pi^{(D-k)}(\text{aff}(F))$. Hence, F_0 is affinely integral. Therefore, $\pi^{(D-k)}(P)$ is fully integral. \square

Lemma 7.3. *Suppose $0 \leq k \leq d \leq D$ and $P \subset V$ is a d -dimensional polytope whose k -faces are all in affinely general position. Let $\mathbf{y} \in \pi^{(D-k)}(P)$.*

- (i) *If \mathbf{y} is an interior point of $\pi^{(D-k)}(P)$, then the slice $\pi_{D-k}(\mathbf{y}, P)$ is a $(d - k)$ -dimensional polytope containing an interior point of P .*
- (ii) *If \mathbf{y} is on the boundary of $\pi^{(D-k)}(P)$, then the slice $\pi_{D-k}(\mathbf{y}, P)$ is one point on the boundary of P . (Note that this case would not happen if $k = 0$ because $\pi^{(D-k)}(P) = V_0$ does not have boundary points.)*

Proof. Let $U = \text{aff}(P)$. Because k -faces of P are in affinely general position, one checks that $\pi^{(D-k)}(U) = V_k$. Therefore, $\pi_{D-k}(\mathbf{y}, U)$ is $(d - k)$ -dimensional, for any $\mathbf{y} \in V_k$.

Since $\pi^{(D-k)}$ is a continuous open map and P is convex, we have that

$$\pi^{(D-k)}(\text{Int}(P)) = \text{Int}(\pi^{(D-k)}(P)).$$

Thus, if \mathbf{y} is an interior point of $\pi^{(D-k)}(P)$, then $\pi_{D-k}(\mathbf{y}, P)$ contains an interior point of P and $\dim(\pi_{D-k}(\mathbf{y}, P)) = \dim(\pi_{D-k}(\mathbf{y}, U)) = d - k$. Now suppose \mathbf{y} is a boundary point of $\pi^{(D-k)}(P)$. Then \mathbf{y} is on a facet, which is a $(k - 1)$ -face, of $\pi^{(D-k)}(P)$. Hence, (ii) follows immediately from Lemma 7.1 with $\ell = m = k$. \square

Corollary 7.4. *Suppose $k < d$ and $P \subset V$ is a d -dimensional rational polytope.*

(i) If all of the k -faces of P are affinely integral, then

$$\sum_{\mathbf{y} \in \pi^{(D-k)}(\Lambda_P)} \text{Vol}_{\Lambda^k \cap \Lambda_{\text{lin}(P)}}(\pi_{D-k}(\mathbf{y}, P)) = \sum_{\mathbf{y} \in \Lambda \cap \text{Int}(\pi^{(D-k)}(P))} \text{Vol}_{\Lambda^k \cap \Lambda_{\text{lin}(P)}}(\pi_{D-k}(\mathbf{y}, P)).$$

(ii) If P is central and all of the k -faces of P are in affinely general position, then

$$\text{SVol}_{\Lambda_P}^k(P) = \sum_{\mathbf{y} \in \pi^{(D-k)}(\Lambda_P) \cap \text{Int}(\pi^{(D-k)}(P))} \text{Vol}_{\Lambda^k \cap \Lambda_P}(\pi_{D-k}(\mathbf{y}, P)).$$

Proof. We prove (ii) first. Since $d - k > 0$, by Lemma 3.11(ii) and Lemma 7.3, for any \mathbf{y} on the boundary of $\pi^{(D-k)}(P)$, we have that $\text{Vol}_{\Lambda^k \cap \Lambda_P}(\pi_{D-k}(\mathbf{y}, P)) = 0$. Thus, (ii) follows.

Now we prove (i). One checks that it is enough to prove (i) under the assumption that P is central. Then (i) follows from (ii), Corollary 3.10(i) and Lemma 3.11(i). \square

The polytope $\pi_{D-k}(\mathbf{y}, P)$ is the intersection of the polytope P with the affine space $\pi_{D-k}(\mathbf{y}, V)$. Hence, faces of $\pi_{D-k}(\mathbf{y}, P)$ are exactly the intersections of faces of P with $\pi_{D-k}(\mathbf{y}, V)$. In other words,

$$\mathcal{F}(\pi_{D-k}(\mathbf{y}, P)) = \{ \pi_{D-k}(\mathbf{y}, F) \mid F \in \mathcal{F}(P) \}. \tag{7.1}$$

The following lemma discuss details of the relation between $\mathcal{F}(\pi_{D-k}(\mathbf{y}, P))$ and $\mathcal{F}(P)$.

Lemma 7.5. *Suppose $P \subset V$ is a d -dimensional polytope whose k -faces are all in affinely general position. Let $\mathbf{y} \in \pi^{(D-k)}(P)$. For any nonempty face F_0 of $\pi_{D-k}(\mathbf{y}, P)$, there exists a face F of P such that*

- (i) $\dim(F) = \dim(F_0) + k$,
- (ii) $F_0 = \pi_{D-k}(\mathbf{y}, F)$.

Furthermore, if $\dim(F_0) > 0$, then such an F is unique and we also have that $\mathbf{y} \in \text{Int}(\pi^{(D-k)}(F))$.

Proof. Suppose $\dim(F_0) = r$. Because of (7.1), there exists a face F of P such that $F_0 = \pi_{D-k}(\mathbf{y}, F)$. We consider

$$\{ F \in \mathcal{F}(P) \mid F_0 = \pi_{D-k}(\mathbf{y}, F) \}. \tag{7.2}$$

We claim that if (7.2) contains one face of dimension less than or equal to k , then $\dim(F_0) = 0$ and (7.2) contains one face of dimension k . Suppose F is a face in (7.2) of dimension less than or equal to k , then F is contained in a k -face F' of P . Because F' is in affinely general position, it follows from Corollary 3.10(iii) that $\pi_{D-k}(\mathbf{y}, F')$ contains at most one point. However, $F_0 = \pi_{D-k}(\mathbf{y}, F) \subset \pi_{D-k}(\mathbf{y}, F')$ and F_0 contains at least one point. We must have that $F_0 = \pi_{D-k}(\mathbf{y}, F')$ is one point and F' is in (7.2).

Suppose $\dim(F_0) = 0$, i.e., F_0 is a point. We want to show that the set (7.2) contains one face of dimension k . We have already seen that this is true if (7.2) contains one face F of dimension less than or equal to k . Suppose all the faces in (7.2) have dimension greater than k , and let

F be one that has the smallest dimension. All the k -faces of F are k -faces of P , thus are in affinely general position. Hence, by Lemma 7.3, we must have that $F_0 = \pi_{D-k}(\mathbf{y}, F)$ is on the boundary of F . Therefore, there exists a proper face F' of F such that $F_0 \in F'$. It is clear that $F_0 = \pi_{D-k}(\mathbf{y}, F')$. However, $\dim(F') < \dim(F)$, which contradicts the choice of F .

Suppose $\dim(F_0) > 0$. We want to show that the set (7.2) contains exactly one face, which has dimension $\dim(F_0) + k$. Let F be a face in (7.2). By our claim, $\dim(F) > k$. Again, all the k -faces of F are in affinely general position. Thus, by Lemma 7.3, we have that $\mathbf{y} \in \text{Int}(\pi^{(D-k)}(F))$ and $F_0 = \pi_{D-k}(\mathbf{y}, F)$ has dimension $\dim(F) - k$. Hence, $\dim(F) = \dim(F_0) + k$. The uniqueness follows from the observation that if $F_1, F_2 \in (7.2)$, then so is $F_1 \cap F_2$. \square

Corollary 7.6. *Suppose $P \subset V$ is a d -dimensional polytope whose k -faces are all in affinely general position. Let $\mathbf{y} \in \text{Int}(\pi^{(D-k)}(P))$.*

- (i) *Suppose $1 \leq r \leq d - k$. The map $\pi_{D-k}(\mathbf{y}, \cdot)$ gives a bijection between the set of $(r + k)$ faces of P satisfying $\mathbf{y} \in \text{Int}(\pi^{(D-k)}(F))$ and the set of r -faces of $\pi_{D-k}(\mathbf{y}, P)$.*
- (ii) *The map $\pi_{D-k}(\mathbf{y}, \cdot)$ gives a bijection between the set $\{F \in \mathcal{F}_{\geq k+1}(P) \mid \mathbf{y} \in \text{Int}(\pi^{(D-k)}(F))\}$ and the set $\mathcal{F}_{\geq 1}(\pi_{D-k}(\mathbf{y}, P))$.*
- (iii) *Suppose P is a rational polytope. If F is a face of P that corresponds under $\pi_{D-k}(\mathbf{y}, \cdot)$ to a face F_0 of $\pi_{D-k}(\mathbf{y}, P)$, then $\Lambda_{\text{lin}(F_0)} = \Lambda^k \cap \Lambda_{\text{lin}(F)}$.*

Proof. (i) follows from Lemma 7.5 and Lemma 7.3(i). (ii) follows from (i) and the fact that P is d -dimensional.

If F is in bijection with F_0 under $\pi_{D-k}(\mathbf{y}, \cdot)$, then $F_0 = \pi_{D-k}(\mathbf{y}, F)$. One checks that $\text{lin}(F_0) = V^k \cap \text{lin}(F)$. Then (iii) follows. \square

The original definitions of general position and integrality of an affine space require the space to sit inside the space spanned by the given lattice. It cannot be applied to slices of a polytope P since they are in the affine spaces $\pi_{D-k}(\mathbf{y}, V)$. Hence, we extend these definitions.

Definition 7.7. Let U be an ℓ -dimensional rational subspace of V . Suppose $\mathfrak{f} = (\mathbf{f}^1, \dots, \mathbf{f}^\ell)$ is a basis of the lattice $U \cap \Lambda$. Let W be an affine space of V such that $\text{lin}(W)$ is a subspace of U .

- (a) We say W is *integral with respect to \mathfrak{f}* if W contains a lattice point and $\text{lin}(W)$ is integral with respect to \mathfrak{f} .
- (b) We say W is *in general position with respect to \mathfrak{f}* if $\text{lin}(W)$ is in general position with respect to \mathfrak{f} .

Similarly, we extend the definitions of *affinely integral*, *k-integral*, *affinely general position*, and *k-general position with respect to \mathfrak{f}* for polytopes.

Lemma 7.8. *Let W be an ℓ -dimensional affine space of V . Suppose $m: 0 \leq m \leq \ell$.*

- (i) *If W is integral, then $\pi_{D-m}(\mathbf{y}, W)$ is integral with respect to $(\mathbf{e}^{m+1}, \dots, \mathbf{e}^D)$ for any $\mathbf{y} \in \Lambda_m$.*
- (ii) *If W is in general position, then $\pi_{D-m}(\mathbf{y}, W)$ is in general position with respect to $(\mathbf{e}^{m+1}, \dots, \mathbf{e}^D)$ for any $\mathbf{y} \in V_m$.*

Proof. We will only prove (i), as (ii) can be proved similarly. By Lemma 3.8, $\text{lin}(W)$ is integral and $W = \text{lin}(W) + \mathbf{z}$ for some $\mathbf{z} \in \Lambda \cap W$. Then $\text{lin}(W)$ is the row space of a matrix of the form (I, J) where I is the $\ell \times \ell$ identity matrix and J is an $\ell \times (D - \ell)$ integer matrix. Let A be the top $m \times D$ submatrix of (I, J) and B be the bottom $(\ell - m) \times D$ submatrix of (I, J) . It is easy to verify that

$$\pi_{D-m}(\mathbf{y}, W) = \text{the row space of } B + ((\mathbf{y} - \pi^{(D-m)}(\mathbf{z}))A + \mathbf{z}).$$

One sees that

$$\text{lin}(\pi_{D-m}(\mathbf{y}, W)) = \text{the row space of } B \subset V^m.$$

Note that the first m columns of B are zeros, which is followed by the $(\ell - m) \times (\ell - m)$ identity matrix and an $(\ell - m) \times (D - \ell)$ integer matrix. Thus, $\text{lin}(\pi_{D-m}(\mathbf{y}, W))$ is integral with respect to $(\mathbf{e}^{m+1}, \dots, \mathbf{e}^D)$. Since $(\mathbf{y} - \pi^{(D-m)}(\mathbf{z}))A + \mathbf{z}$ is a lattice point in W , we conclude that $\pi_{D-m}(\mathbf{y}, W)$ is integral with respect to $(\mathbf{e}^{m+1}, \dots, \mathbf{e}^D)$. \square

Lemma 7.9. *Suppose $0 \leq m \leq \ell \leq d$ and $P \subset V$ is a d -dimensional polytope.*

- (i) *Suppose all the ℓ -faces of P are affinely integral. Then for each \mathbf{y} in $\pi^{(D-m)}(P) \cap \Lambda$, we have that every $(\ell - m)$ -face of $\pi_{D-m}(\mathbf{y}, P)$ is affinely integral with respect to $(\mathbf{e}^{m+1}, \dots, \mathbf{e}^D)$.*
- (ii) *Suppose all the ℓ -faces of P are in affinely general position. Then for each \mathbf{y} in $\pi^{(D-m)}(P)$ we have that every $(\ell - m)$ -face of $\pi_{D-m}(\mathbf{y}, P)$ is in affinely general position with respect to $(\mathbf{e}^{m+1}, \dots, \mathbf{e}^D)$.*

Proof. We only prove (i), and (ii) can be proved similarly. Let F_0 be a face of $\pi_{D-m}(P)$ of dimension $\ell - m$. By Lemma 7.5, there exists a face F of P such that $\dim(F) = \dim(F_0) + m = \ell$ and $F_0 = \pi_{D-m}(\mathbf{y}, F)$. We have that F is affinely integral, i.e., $\text{aff}(F)$ is integral. One checks that $\text{aff}(F_0) = \text{aff}(\pi_{D-m}(\mathbf{y}, F)) = \pi_{D-m}(\mathbf{y}, \text{aff}(F))$, which by Lemma 7.8(i) is integral with respect to $(\mathbf{e}^{m+1}, \dots, \mathbf{e}^D)$. Therefore, F_0 is affinely integral with respect to $(\mathbf{e}^{m+1}, \dots, \mathbf{e}^D)$. \square

Corollary 7.10. *Suppose all the k -faces of P are affinely integral. Then for each $\mathbf{y} \in \pi^{(D-k)}(P) \cap \Lambda$, we have that $\pi_{D-k}(\mathbf{y}, P)$ is an integral polytope.*

Proof. This follows from Lemma 7.9(i) with $m = \ell = k$. \square

8. Proof of Proposition 3.16 and Theorem 1.5

Proof of Proposition 3.16 (and Theorem 1.5). We will prove the proposition by induction on k . The base case $k = 1$ follows from Proposition 6.1. Let $k_0: 2 \leq k_0 < d$. Now we assume that the theorem holds for any $k \leq k_0$, and we will prove the theorem for $k = k_0$. By Corollary 7.4(ii), it is sufficient to show that

$$\text{Vol}_{\Lambda_P}(P) = \sum_{\mathbf{z} \in \pi^{(D-k)}(\Lambda_P) \cap \text{Int}(\pi^{(D-k_0)}(P))} \text{Vol}_{\Lambda^{k_0} \cap \Lambda_P}(\pi_{D-k_0}(\mathbf{z}, P)). \tag{8.1}$$

Since $k_0 \geq 2$, we have that P is an (0) -integral polytope in 1 -general position. Hence, by Proposition 6.1 and Corollary 7.4(ii), we have that

$$\text{Vol}_{\Lambda_P}(P) = \sum_{y \in \pi^{(D-1)}(\Lambda_P) \cap \text{Int}(\pi^{(D-1)}(P))} \text{Vol}_{\Lambda^1 \cap \Lambda_P}(\pi_{D-1}(y, P)).$$

It follows from Lemma 7.9 that the slice $\pi_{D-1}(y, P)$ is $(k_0 - 2)$ -integral and in $(k_0 - 1)$ -general position with respect to $(\mathbf{e}^2, \dots, \mathbf{e}^D)$, for any $y \in \pi^{(D-k)}(\Lambda_P) \cap \text{Int}(\pi^{(D-1)}(P))$. Thus, $\pi_{D-1}(y, P) - y$ is a central polytope in V^1 that is $(k_0 - 2)$ -integral and in $(k_0 - 1)$ -general position with respect to $(\mathbf{e}^2, \dots, \mathbf{e}^D)$. Moreover, the lattice in V^1 is Λ^1 , and the lattice we use to calculate the normalized volume of $\pi_{D-1}(y, P) - y$ is

$$\Lambda^1 \cap \text{lin}(\pi_{D-1}(y, P) - y) = \Lambda^1 \cap \pi_{D-1}(0, \text{lin}(P)) = \Lambda^1 \cap (V^1 \cap \text{lin}(P)) = \Lambda^1 \cap \Lambda_P.$$

Hence, we can apply the induction hypothesis to $\pi_{D-1}(y, P) - y$ and translate the result to $\pi_{D-1}(y, P)$. We then get

$$\text{Vol}_{\Lambda^1 \cap \Lambda_P}(\pi_{D-1}(y, P)) = \sum_{\substack{\mathbf{z} \in \pi^{(D-k_0)}(\Lambda^1 \cap \Lambda_P) + y \\ \mathbf{z} \in \text{Int}(\pi^{(D-k_0)}(\pi_{D-1}(y, P)))}} \text{Vol}_{\Lambda^{k_0} \cap \Lambda_P}(\pi_{D-k_0}(\mathbf{z}, \pi_{D-1}(y, P))).$$

One checks that for any $\mathbf{z} \in V_{k_0}$, we have that $\mathbf{z} \in \pi^{(D-k_0)}(\Lambda^1 \cap \Lambda_P) + y$ if and only if $\mathbf{z} \in \pi^{(D-k_0)}(\Lambda_P)$ and the first coordinate of \mathbf{z} is y . Furthermore, $\mathbf{z} \in \text{Int}(\pi^{(D-k_0)}(\pi_{D-1}(y, P))) = \text{Int}(\pi_{D-1}(y, \pi^{(D-k_0)}(P)))$ if and only if the first coordinate of \mathbf{z} is y , and $\mathbf{z} \in \text{Int}(\pi^{(D-k_0)}(P))$. Also, if $\mathbf{z} \in \pi^{(D-k_0)}(\pi_{D-1}(y, P))$, then $\pi_{D-k_0}(\mathbf{z}, \pi_{D-1}(y, P)) = \pi_{D-k_0}(\mathbf{z}, P)$. Therefore, we have

$$\begin{aligned} \text{Vol}_{\Lambda_P}(P) &= \sum_{y \in \pi^{(D-1)}(\Lambda_P) \cap \text{Int}(\pi^{(D-1)}(P))} \sum_{\substack{\mathbf{z} \in \pi^{(D-k_0)}(\Lambda^1 \cap \Lambda_P) + y \\ \mathbf{z} \in \text{Int}(\pi^{(D-k_0)}(\pi_{D-1}(y, P)))}} \text{Vol}_{\Lambda^{k_0} \cap \Lambda_P}(\pi_{D-k_0}(\mathbf{z}, \pi_{D-1}(y, P))) \\ &= \sum_{y \in \pi^{(D-1)}(\Lambda_P) \cap \text{Int}(\pi^{(D-1)}(P))} \sum_{\substack{\mathbf{z} \in \pi^{(D-k_0)}(\Lambda_P) \cap \text{Int}(\pi^{(D-k_0)}(P)) \\ \pi^{(D-1)}(\mathbf{z}) = y}} \text{Vol}_{\Lambda^{k_0} \cap \Lambda_P}(\pi_{D-k_0}(\mathbf{z}, P)). \end{aligned}$$

Furthermore,

$$\begin{aligned} \pi^{(k_0-1)}(\text{Int}(\pi^{(D-k_0)}(P))) &= \pi^{(k_0-1)}(\pi^{(D-k_0)}(\text{Int}(P))) = \pi^{(D-1)}(\text{Int}(P)) \\ &= \text{Int}(\pi^{(D-1)}(P)), \end{aligned}$$

which means that $\text{Int}(\pi^{(D-1)}(P))$ contains all possible first coordinates of points in $\text{Int}(\pi^{(D-k_0)}(P))$.

Therefore, (8.1) follows. \square

9. Proof of Theorem 1.3

We first recall a result which we will use to prove Theorem 1.3.

Let $P \subset V$ be a nonempty polyhedron and let $\mathbf{v} \in P$ be a point. We define the *feasible cone of P at \mathbf{v}* by

$$\text{fcone}(P, \mathbf{v}) = \{\mathbf{y} \mid \mathbf{v} + \epsilon\mathbf{y} \in P \text{ for some } \epsilon > 0\}.$$

For any face F of P and any two interior points \mathbf{v}, \mathbf{w} of F , one checks that $\text{fcone}(P, \mathbf{v}) = \text{fcone}(P, \mathbf{w})$. Therefore, we can define the *feasible cone of P along F* by

$$\text{fcone}(P, F) := \text{fcone}(P, \mathbf{v}), \quad \text{for some (equivalently all) } \mathbf{v} \in \text{Int}(F).$$

In the context of toric varieties, Danilov asked in [6] whether there exist numbers $\alpha(P, F)$ depending only on $\text{fcone}(P, F)$ such that for an integral polytope P ,

$$i(P) = \sum_{F \in \mathcal{F}_{\geq 0}(P)} \alpha(P, F) \text{Vol}_{\Lambda_{\text{lin}}(F)}(F). \tag{9.1}$$

This was first proved to be true by McMullen [11]. Morelli [12] was the first to supply an explicit and computationally efficient way to choose $\alpha(P, F)$. In [13], Pommersheim and Thomas gave a canonical construction of $\alpha(P, F)$ from the Todd class of a toric variety. In 2005, Berline and Vergne [5] gave the first algorithmic construction of $\alpha(P, F)$ in the primary space. We follow the notation of [3].

For any integral polytope P , since $\alpha(P, F)$ depends only on $\text{fcone}(P, F)$ and $\text{fcone}(P, F) = \text{fcone}(mP, mF)$, we immediately have that

$$i(P, m) = \sum_{F \in \mathcal{F}_{\geq 0}(P)} \alpha(P, F) \text{Vol}_{\Lambda_{\text{lin}}(F)}(F) m^{\dim(F)}. \tag{9.2}$$

Therefore, the coefficient of m^ℓ in the Ehrhart polynomial of P is given by the sum of $\alpha(P, F) \text{Vol}_{\Lambda_{\text{lin}}(F)}(F)$ over all faces of dimension ℓ . In particular, because the constant term of $i(P, m)$ is 1, and the normalized volume of a point is 1, we have that

$$\sum_{\mathbf{v} \in \text{vert}(P)} \alpha(P, \mathbf{v}) = 1.$$

Applying the above identity to (9.1) and (9.2), we get the following corollary.

Corollary 9.1. *If P is an integral polytope, then*

$$i(P) - 1 = \sum_{F \in \mathcal{F}_{\geq 1}(P)} \alpha(P, F) \text{Vol}_{\Lambda_{\text{lin}}(F)}(F),$$

and

$$i(P, m) - 1 = \sum_{F \in \mathcal{F}_{\geq 1}(P)} \alpha(P, F) \text{Vol}_{\Lambda_{\text{lin}(F)}}(F) m^{\dim(F)}.$$

Lemma 9.2. *Suppose $P \subset V$ is a d -dimensional polytope whose k -faces are all affinely integral. Let F be a face of P of dimension greater than k . Then*

- (i) $\pi_{D-k}(\mathbf{y}, F)$ is a $(\dim(F) - k)$ -face of $\pi_{D-k}(\mathbf{y}, P)$, for any lattice point \mathbf{y} in $\text{Int}(\pi^{(D-k)}(F))$.
- (ii) $\alpha(\pi_{D-k}(\mathbf{y}, P), \pi_{D-k}(\mathbf{y}, F))$ is invariant under different choices of lattice points \mathbf{y} in $\text{Int}(\pi^{(D-k)}(P))$.

Proof. (i) follows from Corollary 7.6.

By Corollary 7.10, $\pi_{D-k}(\mathbf{y}, P)$ is integral. One checks that $\text{fcone}(\pi_{D-k}(\mathbf{y}, P), \pi_{D-k}(\mathbf{y}, F))$ is exactly $\text{fcone}(P, F) \cap V^k$, which is independent of the choice of \mathbf{y} . Then (ii) follows. \square

Given this lemma, we are able to give the following definition.

Definition 9.3. *Suppose $P \subset V$ is a d -dimensional polytope whose k -faces are all affinely integral. Let F be a face of P of dimension greater than k . We define*

$$\beta_k(P, F) := \alpha(\pi_{D-k}(\mathbf{y}, P), \pi_{D-k}(\mathbf{y}, F)),$$

for some (equivalently all) $\mathbf{y} \in \Lambda \cap \text{Int}(\pi^{(D-k)}(F))$.

In the following lemma, we give formulas connecting numbers of lattice points of slices to $\beta_k(P, F)$'s and normalized volumes of faces F of dimension at least $k + 1$.

Lemma 9.4. *Suppose $0 \leq k < d \leq D$ and $P \subset V$ is a k -integral d -dimensional polytope. For any positive integer m , we have that*

$$\sum_{\mathbf{y} \in \Lambda \cap \pi^{(D-k)}(P)} (i(\pi_{D-k}(\mathbf{y}, P), m) - 1) = m^{-k} \sum_{F \in \mathcal{F}_{\geq k+1}(P)} \beta_k(P, F) \text{Vol}_{\Lambda_{\text{lin}(F)}}(F) m^{\dim(F)}. \tag{9.3}$$

Hence,

$$\sum_{\mathbf{y} \in \Lambda \cap \pi^{(D-k)}(P)} (i(\pi_{D-k}(\mathbf{y}, P)) - 1) = \sum_{F \in \mathcal{F}_{\geq k+1}(P)} \beta_k(P, F) \text{Vol}_{\Lambda_{\text{lin}(F)}}(F). \tag{9.4}$$

In particular, if $k = d - 1$, then

$$\sum_{\mathbf{y} \in \Lambda \cap \pi^{(D-(d-1))}(P)} (i(\pi_{D-(d-1)}(\mathbf{y}, P)) - 1) = \text{Vol}_{\Lambda_{\text{lin}(P)}}(P). \tag{9.5}$$

Proof. If $k = 0$, then (9.3) follows from Corollary 9.1 and the observation that the left-hand side of (9.3) is $i(P, m) - 1$, and $\beta_0(P, F) = \alpha(P, F)$. Hence, we can assume $k \geq 1$. By Corollary 7.10, $\pi_{D-k}(\mathbf{y}, P)$ is integral, for any $\mathbf{y} \in \Lambda \cap \pi^{(D-k)}(P)$. If \mathbf{y} is on the boundary of $\pi^{(D-k)}(P)$, by Lemma 7.3(ii), $\pi_{D-k}(\mathbf{y}, P)$ is just one point. Therefore, $i(\pi_{D-k}(\mathbf{y}, P), m) - 1 = 0$. Using this and Corollary 9.1, we have that the left-hand side of (9.3) is equal to

$$\sum_{\mathbf{y} \in \Lambda \cap \text{Int}(\pi^{(D-k)}(P))} \sum_{F_0 \in \mathcal{F}_{\geq 1}(\pi_{D-k}(\mathbf{y}, P))} \alpha(\pi_{D-k}(\mathbf{y}, P), F_0) \text{Vol}_{\Lambda_{\text{lin}}(F_0)}(F_0) m^{\dim(F_0)}. \tag{9.6}$$

However, by Corollary 7.6, we can rewrite (9.6) as

$$\begin{aligned} & \sum_{\mathbf{y} \in \Lambda \cap \text{Int}(\pi^{(D-k)}(P))} \sum_{\substack{F \in \mathcal{F}_{\geq k+1}(P) \\ \mathbf{y} \in \text{Int}(\pi^{(D-k)}(F))}} [\alpha(\pi_{D-k}(\mathbf{y}, P), \pi_{D-k}(\mathbf{y}, F)) \\ & \qquad \qquad \qquad \times \text{Vol}_{\Lambda^k \cap \Lambda_{\text{lin}}(F)}(\pi_{D-k}(\mathbf{y}, F)) m^{\dim(F)-k}] \\ &= \sum_{\mathbf{y} \in \Lambda \cap \text{Int}(\pi^{(D-k)}(P))} \sum_{\substack{F \in \mathcal{F}_{\geq k+1}(P) \\ \mathbf{y} \in \text{Int}(\pi^{(D-k)}(F))}} \beta_k(P, F) \text{Vol}_{\Lambda^k \cap \Lambda_{\text{lin}}(F)}(\pi_{D-k}(\mathbf{y}, F)) m^{\dim(F)-k}. \end{aligned}$$

However, for any face $F \in \mathcal{F}_{\geq k+1}(P)$, it follows from the hypothesis that P is k -integral that F is k -integral as well. By Corollary 7.2, we have that $\dim(\pi^{(D-k)}(F)) = k = \dim(\pi^{(D-k)}(P))$. Hence, $\text{Int}(\pi^{(D-k)}(F)) \subset \text{Int}(\pi^{(D-k)}(P))$. Therefore, we can simplify the above formula further and conclude that

$$\begin{aligned} & \sum_{\mathbf{y} \in \Lambda \cap \pi^{(D-k)}(P)} (i(\pi_{D-k}(\mathbf{y}, P), m) - 1) \\ &= \sum_{\substack{F \in \mathcal{F}_{\geq k+1}(P) \\ \mathbf{y} \in \Lambda \cap \text{Int}(\pi^{(D-k)}(F))}} \beta_k(P, F) \text{Vol}_{\Lambda^k \cap \Lambda_{\text{lin}}(F)}(\pi_{D-k}(\mathbf{y}, F)) m^{\dim(F)-k} \\ &= m^{-k} \sum_{F \in \mathcal{F}_{\geq k+1}(P)} \beta_k(P, F) m^{\dim(F)} \sum_{\mathbf{y} \in \Lambda \cap \text{Int}(\pi^{(D-k)}(F))} \text{Vol}_{\Lambda^k \cap \Lambda_{\text{lin}}(F)}(\pi_{D-k}(\mathbf{y}, F)). \end{aligned}$$

Since any face $F \in \mathcal{F}_{\geq k+1}(P)$ is k -integral, and any k -integral polytope is $(k - 1)$ -integral and in k -general position. Therefore, by Theorem 1.5 and Corollary 7.4(i), the identity (9.3) holds. (9.4) is obtained from (9.3) by plugging in $m = 1$.

Finally, in the case of $k = d - 1$, note that the only face F in $\mathcal{F}_{\geq d}(P)$ is P , so (9.5) would follow from (9.4) given the identity

$$\beta_{d-1}(P, P) = \alpha(\pi_{D-(d-1)}(\mathbf{y}, P), \pi_{D-(d-1)}(\mathbf{y}, P)) = 1.$$

However, in the construction of $\alpha(P, F)$ in [5], $\alpha(Q, Q)$ is always 1 for any rational polytope. So we have the above identity. \square

Remark 9.5. It is possible to give a proof of (9.5) that makes no use of the results of [5]. In fact, we only need the simple fact that the normalized volume of any 1-dimensional integral polytope is equal to the number of lattice points minus 1, and then we can prove (9.5) with arguments similar to those we used in proving (9.3) (but without involving $\beta_k(P, F)$).

Proposition 9.6. *Suppose $0 \leq k < d \leq D$ and $P \subset V$ is a k -integral d -dimensional polytope. Then*

$$i(P) = i(\pi^{(D-k)}(P)) + \sum_{F \in \mathcal{F}_{\geq k+1}(P)} \beta_k(P, F) \text{Vol}_{\Lambda_{\text{lin}(F)}}(F). \tag{9.7}$$

In particular, if $k = d - 1$, then

$$i(P) = i(\pi^{(D-(d-1))}(P)) + \text{Vol}_{\Lambda_{\text{lin}(P)}}(P). \tag{9.8}$$

Proof. It is clear that

$$\begin{aligned} i(P) &= \sum_{\mathbf{y} \in \Lambda \cap \pi^{(D-k)}(P)} i(\pi_{D-k}(\mathbf{y}, P)) \\ &= i(\pi^{(D-k)}(P)) + \sum_{\mathbf{y} \in \Lambda \cap \pi^{(D-k)}(P)} (i(\pi_{D-k}(\mathbf{y}, P)) - 1). \end{aligned}$$

Then the proposition follows from Lemma 9.4. \square

Using (9.8), we are able to generalize the results in [8–10] to fully integral polytopes:

Corollary 9.7. *Suppose $0 \leq d \leq D$ and $P \subset V$ is a d -dimensional fully integral polytope. Then*

$$i(P) = \sum_{j=0}^d \text{Vol}_{\Lambda_j}(\pi^{(D-j)}(P)). \tag{9.9}$$

Hence, the Ehrhart polynomial of P is given by

$$i(P, m) = \sum_{j=0}^d \text{Vol}_{\Lambda_j}(\pi^{(D-j)}(P)) m^j. \tag{9.10}$$

Proof. It is clear that the corollary holds for $d = 0$. Now we assume $d \geq 1$.

P is fully integral, so P is $(d - 1)$ -integral and $\text{aff}(P)$ is integral. Therefore, by Proposition 9.6 and Corollary 3.10(ii), we have that $i(P) = i(\pi^{(D-(d-1))}(P)) + \text{Vol}_{\Lambda_{\text{lin}(P)}}(P)$, and $\pi^{(D-d)}$ induces a bijection between $\text{aff}(P) \cap \Lambda$ and Λ_d . Because $\text{aff}(P)$ contains a lattice point, $\text{aff}(P) \cap \Lambda$ and $\Lambda_{\text{lin}(P)}$ only differ by a lattice point. Hence, $\pi^{(D-d)}$ induces a bijection between $\Lambda_{\text{lin}(P)}$ and Λ_d . Therefore, we conclude that

$$i(P) = i(\pi^{(D-(d-1))}(P)) + \text{Vol}_{\Lambda_d}(\pi^{(D-d)}(P)).$$

However, it follows from Corollary 7.2 that $\pi^{(D-(d-1))}(P)$ is a $(d-1)$ -dimensional fully integral polytope. Therefore, we obtain (9.9) by recursively applying the above identity.

Finally, by Lemma 3.12, the dilation mP of P is fully integral, for any positive integer m . Thus, (9.10) follows. \square

Proof of Theorem 1.3. If $k = d$, the theorem follows from (9.10). Now we assume $k < d$. Let m be a positive integer. By Lemma 3.12, mP is a k -integral polytope. For any $F \in \mathcal{F}_{\geq k+1}(P)$, one checks that $\beta_k(mP, mF) = \beta_k(P, F)$ and $\text{lin}(mF) = \text{lin}(F)$. Hence, by Proposition 9.6, we have that

$$\begin{aligned} i(P, m) &= i(\pi^{(D-k)}(P), m) + \sum_{F \in \mathcal{F}_{\geq k+1}(P)} \beta_k(P, F) \text{Vol}_{\Lambda_{\text{lin}(F)}}(mF) \\ &= i(\pi^{(D-k)}(P), m) + \sum_{F \in \mathcal{F}_{\geq k+1}(P)} \beta_k(P, F) \text{Vol}_{\Lambda_{\text{lin}(F)}}(F) m^{\dim(F)}. \end{aligned}$$

By Corollary 7.2, $\pi^{(D-k)}(P)$ is a k -dimensional fully integral polytope. Therefore, we conclude our theorem using (9.3) and (9.10). \square

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