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Symmetry classes of alternating-sign matrices under one roof

By Greg Kuperberg*

Abstract

In a previous article [22], we derived the alternating-sign matrix (ASM) theorem from the Izergin-Korepin determinant [12], [19], [13] for a partition function for square ice with domain wall boundary. Here we show that the same argument enumerates three other symmetry classes of alternating-sign matrices: VSASMs (vertically symmetric ASMs), even HTSASMs (half-turn-symmetric ASMs), and even QTSASMs (quarter-turn-symmetric ASMs). The VSASM enumeration was conjectured by Mills; the others by Robbins [30]. We introduce several new types of ASMs: UASMs (ASMs with a U-turn side), UUASMs (two U-turn sides), OSASMs (off-diagonally symmetric ASMs), OOSASMs (off-diagonally, off-antidiagonally symmetric), and UOSASMs (off-diagonally symmetric with U-turn sides). UASMs generalize VSASMs, while UUASMs generalize VHSASMs (vertically and horizontally symmetric ASMs) and another new class, VHPASMs (vertically and horizontally perverse). OSASMs, OOSASMs, and UOSASMs are related to the remaining symmetry classes of ASMs, namely DSASMs (diagonally symmetric), DASASMs (diagonally, anti-diagonally symmetric), and TSASMs (totally symmetric ASMs). We enumerate several of these new classes, and we provide several 2-enumerations and 3-enumerations.

Our main technical tool is a set of multi-parameter determinant and Pfaffian formulas generalizing the Izergin-Korepin determinant for ASMs and the Tsuchiya determinant for UASMs [37]. We evaluate specializations of the determinants and Pfaffians using the factor exhaustion method.

1. Introduction

An *alternating-sign matrix* (or *ASM*) is a matrix with entries 1, 0, and -1, such that the nonzero entries alternate in sign in each row and column, and such that the first and last nonzero entry in each row and column is 1.

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Mills, Robbins, and Rumsey conjectured the following formula for the number of ASMs of order n:

THEOREM 1 (Zeilberger). *There are*

$$A(n)=\frac{1!4!7!\dots(3n-2)!}{n!(n+1)!(n+2)!\dots(2n-1)!}$$

$n\times n$ ASMs.

In 1995, two proofs of the alternating-sign matrix theorem appeared. The first proof, by Zeilberger, showed that ASMs are equinumerous with totally symmetric, self-complementary plane partitions [38]. The second proof relied on the equivalence between ASMs and square ice, and on a determinant formula found by Izergin in work with Korepin [12], [13], [19] for a square ice partition function [22]. The Izergin-Korepin determinant, in turn, depends crucially on the Yang-Baxter equation.

In this article we extend our previous argument to three other symmetry classes of ASMs:

THEOREM 2. *The number of* $n\times n$ ASMs *is given by*

$$A(n)=(-3)^{\binom{n}{2}}\prod_{i,j}\frac{3(j-i)+1}{j-i+n}.$$

The number of $2n+1\times 2n+1$ *vertically symmetric* ASMs (VSASMs) *is given by*

$$A_{\mathrm{V}}(2n+1)=(-3)^{n^2}\prod_{\substack{i,j\le 2n+1\\ 2\mid j}}\frac{3(j-i)+1}{j-i+2n+1}.$$

The number of $2n\times 2n$ *half-turn symmetric* ASMs (HTSASMs) *is given by*

$$\frac{A_{\mathrm{HT}}(2n)}{A(n)}=(-3)^{\binom{n}{2}}\prod_{i,j}\frac{3(j-i)+2}{j-i+n}.$$

The number of $4n\times 4n$ *quarter-turn symmetric* ASMs (QTSASMs) *is given by*

$$A_{\mathrm{QT}}(4n)=A_{\mathrm{HT}}(2n)A(n)^2.$$

In the statement of Theorem 2 and throughout this article, subscripts and products range from 1 to n unless otherwise specified.

The formulas in Theorem 2 were conjectured by Robbins [30] except for the VSASM case, which is due to Mills [31]. Robbins also conjectured formulas for several other types of ASMs, which we discuss in Section 7. The most interesting property of all of these enumerations, both proven and conjectured, is that they are *round*, meaning a product of small factors. Not all types of

ASMs come in round numbers. For example, Robbins found that the number of totally symmetric ASMs (TSASMs) is probably not round. All of the round integers that we compute come from roundness of polynomials, an even stronger property which we discuss in Section 5.

Our main technical tool is a set of determinant and Pfaffian formulas for ASM partition functions Theorem 10. Theorem 10 could be as important as the enumerations which follow from it, but it is too complicated to state in summary here. The formulas include the Izergin-Korepin determinant and a determinant due to Tsuchiya [37]. They imply a number of interesting divisibilities that generalize several that were found experimentally by Robbins [30].

$$\begin{pmatrix} 0 & 0 & 0 & + & 0 & 0 & 0 \\ 0 & 0 & + & - & + & 0 & 0 \\ + & 0 & - & + & - & 0 & + \\ 0 & 0 & + & - & + & 0 & 0 \\ 0 & + & - & + & - & + & 0 \\ 0 & 0 & + & - & + & 0 & 0 \\ 0 & 0 & 0 & + & 0 & 0 & 0 \end{pmatrix}$$

Figure 1. An VSASM with x-weight x^2.

Our arguments for Theorem 2 apply to the x-enumeration of some of the symmetry classes when $x = 2$ and $x = 3$. In the x-enumeration, the weight of a symmetric ASM is x^n if n of the orbits of the entries under symmetry are -1, excluding any -1s that are forced by symmetry. An example is given in Figure 1. (In the figures, we use $+$ and $-$ for 1 and -1.) Note that all of the 2-enumerations other than that of QTSASMs are known by other methods [9], [10], [30], [4], [5], [15], [14], [32], [26], [27], [21]. Our results are the following:

THEOREM 3. *The* 2- *and* 3-*enumerations of* ASMs *and* VSASMs *are given by*

$$\begin{aligned} A(n;2) &= 2^{\binom{n}{2}} \\ A(n;3) &= \frac{3^{n^2-n}}{2^{n^2-n}} \prod_{\substack{i,j \\ 2\nmid j-i}} \frac{3(j-i)+1}{3(j-i)} \\ A_{\mathrm{V}}(2n+1;2) &= 2^{n^2-n} \\ A_{\mathrm{V}}(2n+1;3) &= \frac{3^{2n^2-n}}{2^{2n^2+n}} \prod_{\substack{i,j \le 2n+1 \\ 2\nmid i, 2\mid j}} \frac{3(j-i)+1}{3(j-i)}. \end{aligned}$$

The 2-enumerations of even-sized HTSASMs *and* QTSASMs *are given by*

$$
\begin{aligned}
A_{\mathrm{HT}}(2n;2,1) &= 2^{n^2} \prod_{\substack{i,j \\ 2\nmid j-i}} \frac{2(j-i)+1}{2(j-i)} \\
A_{\mathrm{QT}}(4n;2) &= (-1)^{\binom{n}{2}} 2^{2n^2-n} \prod_{i,j} \frac{4(j-i)+1}{j-i+n}.
\end{aligned}
$$

We will consider the y-weight of a $2n \times 2n$ HTSASM, which is y^k if the HTSASM has k nonzero entries in the upper left quadrant. This yields the (x,y)-enumeration of HTSASMs, which is round when y is -1 and x is 1 or 3. Although the enumeration of VHSASMs remains open, we will enumerate and 3-enumerate a closely related class, VHPASMs [1] (vertically and horizontally perverse). A VHPASM has dimensions $4n+1 \times 4n+3$ for some integer n. It satisfies the alternating-sign condition and it has the same symmetries as a VHSASM, except that the central entry $(*)$ has the opposite sign when read horizontally as when read vertically. The simplest VHPASM is given in Figure 2.

$$
\begin{pmatrix}
0 & 0 & 0 & + & 0 & 0 & 0 \\
0 & + & 0 & - & 0 & + & 0 \\
+ & - & + & * & + & - & + \\
0 & + & 0 & - & 0 & + & 0 \\
0 & 0 & 0 & + & 0 & 0 & 0
\end{pmatrix}
$$

Figure 2. The simplest VHPASM.

VHPASMs come from consideration of two other types of ASMs with U-turn boundary called UASMs and UUASMs[2]. Figure 3 shows an example of a UASM. As the example indicates, a UASM is vertically just like an ASM. Horizontally the signs alternate if we read the $2k-1$st row from left to right, and then continue to alternate if we read the $2k^{\text{th}}$ row from right to left. Likewise both the columns and the rows of a UUASM are like the rows of a UASM. We define the x-weight of a UASM or a UUASM be the number of -1s, as before. We define the y-weight of a UASM to be y^k if k of the U-turns are oriented upward in the corresponding square ice state. We define the y-weight of a UUASM the same way using the U-turns on the right, and define the z-weight of a UUASM to be z^k if k of the U-turns on the top are oriented to the right. Thus we can consider the (x,y)-enumeration of UASMs and the

[1] Also known as β-ASMs, since their boundary conditions are incompatible with VHS.

[2] Also known as Unix-to-Unix ASMs.

(x, y, z)-weight of UUASMs.

$$\begin{pmatrix} 0 & 0 & + \\ 0 & + & - \\ + & - & 0 \\ 0 & 0 & + \\ 0 & + & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Figure 3. A UASM.

Finally we will consider OSASMs (off-diagonally symmetric ASMs, i.e., symmetric ASMs with a null diagonal), OOSASMs (off-diagonally, off-antidiagonally symmetric), and UOSASMs (off-antidiagonally symmetric UUASMs). UOSASMs include as a special case those TSASMs (totally symmetric ASMs with 0s on the diagonals except in the center). As with UASMs, the y-weight of a UOSASM is y^k if k of the U-turns on the top are oriented to the right. By contrast, the y-weight of an OOSASM is y^k if there are $2k$ more 1s than -1s in the upper left quadrant. In the statement of the theorem we index a given type of ASM by the length of one of its rows, counting the length twice if the row takes a U-turn, and we include x-, y-, and z-weight where applicable. For example $A_{\mathrm{UO}}(8n; x, y)$ is the weighted number of $4n \times 4n$ UOSASMs.

THEOREM 4. *There exist polynomials satisfying the equations*

$$\begin{aligned}
A(2n; x) &= 2A_{\mathrm{V}}(2n+1; x)\widetilde{A}_{\mathrm{V}}(2n; x) \\
A(2n+1; x) &= A_{\mathrm{V}}(2n+1; x)\widetilde{A}_{\mathrm{V}}(2n+2; x) \\
A_{\mathrm{U}}(2n; x, y) &= (y+1)^n A_{\mathrm{V}}(2n+1; x) \\
A_{\mathrm{HT}}(2n; x, \pm 1) &= A(n; x) A^{(2)}_{\mathrm{HT}}(2n; x, \pm 1) \\
A_{\mathrm{UU}}(4n; x, y, z) &= A_{\mathrm{V}}(2n+1; x) A^{(2)}_{\mathrm{UU}}(4n; x, y, z) \\
A_{\mathrm{OO}}(4n; x, y) &= A_{\mathrm{O}}(2n; x) A^{(2)}_{\mathrm{OO}}(4n; x, y) \\
A_{\mathrm{QT}}(4n; x) &= A^{(1)}_{\mathrm{QT}}(4n; x) A^{(2)}_{\mathrm{QT}}(4n; x) \\
A_{\mathrm{UO}}(8n; x, y) &= A^{(1)}_{\mathrm{UO}}(8n; x) A^{(2)}_{\mathrm{UO}}(8n; x, y) \\
A^{(2)}_{\mathrm{HT}}(4n; x, 1) &= A^{(2)}_{\mathrm{UU}}(4n; x, 1, 1)\widetilde{A}^{(2)}_{\mathrm{UU}}(4n; x) \\
A^{(2)}_{\mathrm{HT}}(4n+2; x, 1) &= 2A^{(2)}_{\mathrm{UU}}(4n; x, 1, 1)\widetilde{A}^{(2)}_{\mathrm{UU}}(4n+4; x) \\
A^{(2)}_{\mathrm{HT}}(4n; x, -1) &= (-x)^n A^{(1)}_{\mathrm{QT}}(4n; x)^2 \\
A^{(2)}_{\mathrm{OO}}(8n; x, -1) &= (-x)^n A^{(1)}_{\mathrm{UO}}(8n; x, 1)\widetilde{A}^{(1)}_{\mathrm{UO}}(8n; x, 1).
\end{aligned}$$

Many of the factorizations in Theorem 4 were conjectured experimentally by David Robbins [30]; the formula for $A_{\mathrm{U}}(2n;x,y)$ was conjectured by Cohn and Propp [6].

THEOREM 5. *The generating functions in Theorem 4 have the following special values.*

$$\begin{aligned}
A_{\mathrm{O}}(2n) &= A_{\mathrm{V}}(2n+1) \\
A^{(2)}_{\mathrm{UU}}(4n;1,1,1) &= \frac{(-3)^{n^2}}{2^{n(n-1)}} \prod_{\substack{i,j \le 2n+1 \\ 2|j}} \frac{3(j-i)+2}{j-i+2n+1} \\
A^{(2)}_{\mathrm{UU}}(4n;2,1,1) &= 2^{n(n+2)} \prod_{\substack{i,j \le 2n+1 \\ 2\nmid i, 2|j}} \frac{2(j-i)+1}{2(j-i)} \\
A^{(2)}_{\mathrm{VHP}}(4n+2;1) &= A_{\mathrm{V}}(2n+1) \\
A^{(1)}_{\mathrm{QT}}(4n) &= A(n)^2 \\
A^{(1)}_{\mathrm{QT}}(4n;2) &= 2^{n(n-1)} \prod_{i,j} \frac{4(j-i)+1}{j-i+n} \\
A^{(1)}_{\mathrm{QT}}(4n;3) &= 3^{\binom{n}{2}} A(n) \\
A^{(1)}_{\mathrm{UO}}(8n) &= A_{\mathrm{V}}(2n+1)^2 \\
A^{(2)}_{\mathrm{UO}}(8n) &= A_{\mathrm{UU}}(4n).
\end{aligned}$$

(Other identities, for example that

$$A^{(2)}_{\mathrm{QT}}(4n) = A_{\mathrm{HT}}(2n),$$

are implied by combining Theorems 2, 3, 4, and 5, although such combinations do not always reflect the logic of the proofs.)

We will analyze all of the classes ASMs in parallel with ordinary unrestricted ASMs. Along the way we will correct an error in the 3-enumeration of ASMs in Reference 22 originally found by Robin Chapman.

Acknowledgments. The present work began with the mistake found in Reference 22 by Robin Chapman and with the Tsuchiya determinant [37], which is the UASM case of Theorem 10 and which was brought to the author's attention by Jim Propp. We would like to thank Vladimir Korepin, Robin Chapman, and Jim Propp more generally for their attention to the author's work. We would also like to thank Christian Krattanthaler, Soichi Okada, and Paul Zinn-Justin for their interest and for finding mistakes in the first draft.

Finally we would like to acknowledge works by Bressoud [1], Bressoud and Propp [2], Robbins [31], and Zeilberger [38] for spurring the author's interest in alternating-sign matrices.

Mathematical experiments in Maple [25] were essential at every stage of this work. This article is typeset using REVTeX 4 [29] and PSTricks [28].

2. Square ice

If G is a tetravalent graph, an *ice state* (also called a *six-vertex state*) of G is an orientation of the edges such that two edges enter and leave every tetravalent vertex. In particular if G is locally a square grid, then the set of ice states is called *square ice* [24]. More generally G may also have some univalent vertices, which are called *boundary*, and restrictions on the orientations of the boundary edges are called *boundary conditions*.

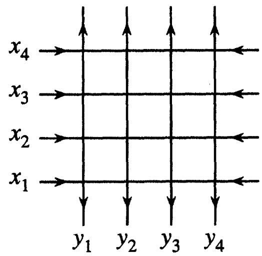


Figure 4. Square ice with domain wall boundary.

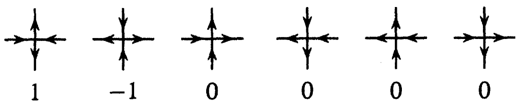


Figure 5. Replacing square ice with alternating-sign entries.

For example, a finite square region of square ice can have *domain wall boundary*, defined as in at the sides and out at the top and bottom, as in Figure 4. These boundary conditions were first considered by Korepin [18], [12], [19]. A square ice state on this region yields a matrix if we replace each vertex by a number according Figure 5. It is easy to check that this transformation is a bijection between square ice with domain wall boundary and alternating-sign matrices [22], [10].

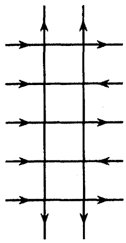


Figure 6. Square ice with M boundary.

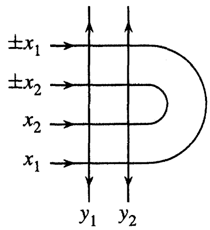


Figure 7. Square ice with HT boundary.

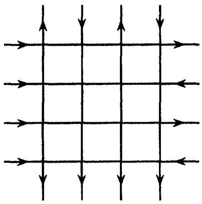


Figure 8. Square ice with VH boundary.

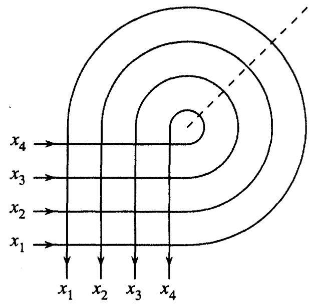


Figure 9. Square ice with QT boundary.

There are also easy bijections from ice states of the graphs in Figures 6–9 to the sets of VSASMs, VHSASMs, even HTSASMs, and even QTSASMs. (The labels in these figures will be used later.) The dashed line in the QTSASM graph means that the orientation of an edge reverses as it crosses the line. The HTSASM and QTSASM graphs are obtained by quotienting the unrestricted ASM graph by the symmetry. The median of a $2n+1 \times 2n+1$ VSASM is always the same, so we can delete it and consider the alternating-sign patterns on the left half. The deleted median then produces the alternating boundary in 6. Likewise we can quarter a VHSASM by deleting both medians, which produces two alternating sides.

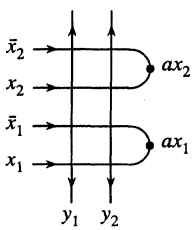


Figure 10. Square ice with U boundary.

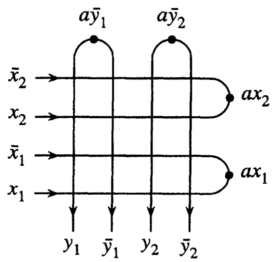


Figure 11. Square ice with UU boundary.

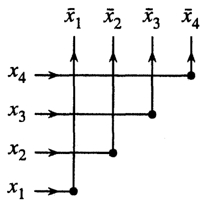


Figure 12. Square ice with O boundary

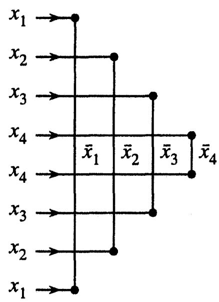


Figure 13. Square ice with OO boundary.

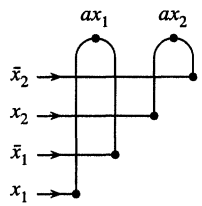


Figure 14. Square ice with UO boundary.

Finally the square ice grids corresponding to UASMs, UUASMs, OSASMs, OOSASMs, and UOSASMs are shown in Figures 10–14. The last three grids have right-angled divalent vertices; we require the orientations of a square ice state to either be both in or both out at these vertices. In contrast at the U-turn vertices one edge must point in and one must point out.

3. Local concerns

Throughout the article we assume the following abbreviations:

$$\begin{aligned} \bar{x} &= x^{-1} \\ \sigma(x) &= x - \bar{x} \\ \alpha(x) &= \sigma(ax)\sigma(a\bar{x}). \end{aligned}$$

(As we discuss below, a is a global parameter that need not appear as an explicit argument of α.)

We will consider a class of multiplicative weights for symmetric ASMs. By a multiplicative weight we mean that the weight of some object is the product of the weights of its parts. In statistical mechanics, multiplicative weights are called *Boltzmann weights*, and the total weight of all objects is called a *partition function*. Figure 15 shows the weights that we will use for the six possible states of a vertex. The figure also shows the weights for U-turns and corners that are labelled with a dot; bare edges and curves have the trivial weight 1. The vertex weights are called an *R-matrix* and the U-turn and corner weights are called *K-matrices.* Vertex and U-turn weights depend on a parameter x (the *spectral parameter*) which may be different for different vertices or U-turns, so we will label sites by the value of x. The weights also depend on three parameters a, b, and c which will be the same for all elements of any single square ice grid, so these parameters do not appear as labels.

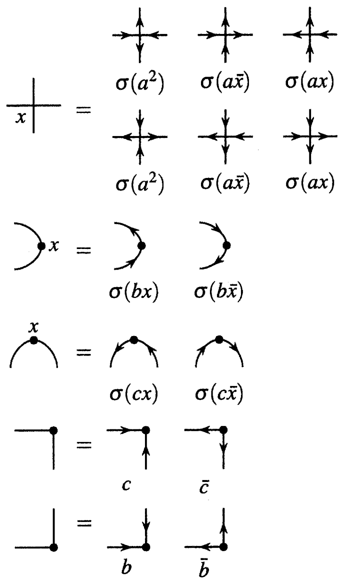
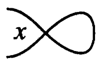
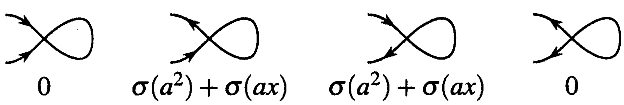


Figure 15. Weights for vertices and U-turns.

We will use a graph with labelled vertices as a notation for its corresponding partition function. If the graph has unoriented boundary edges, then the partition function is also interpreted as a function of the orientations of the edges. On the other hand, our definitions imply that we sum over the orientations of internal edges. For example, the graph

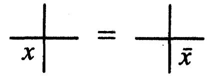


denotes the following function on the set of four orientations of the boundary:

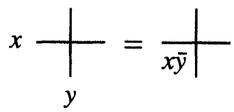


In this notation a vertex is not quite invariant under rotation by 90 degrees, so the meaning of a label depends on the quadrant in which it appears.

The following relation holds:



As a further abbreviation, if we label two lines of a graph that cross at an unlabelled vertex, the spectral parameter is set to their ratio:



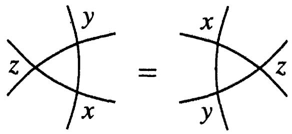
The labelled graphs in Figures 4, 6, 7, 9, 10, 11, 12, 13, and 14 then represent the partition functions

$$\begin{array}{cccc} Z(n;\vec{x},\vec{y}) & Z^{\pm}_{\mathrm{HT}}(n;\vec{x},\vec{y}) & Z_{\mathrm{U}}(n;\vec{x},\vec{y}) & Z_{\mathrm{UU}}(n;\vec{x},\vec{y}) \\ Z_{\mathrm{QT}}(n;\vec{x}) & Z_{\mathrm{O}}(n;\vec{x}) & Z_{\mathrm{OO}}(n;\vec{x}) & Z_{\mathrm{UO}}(n;\vec{x})\ . \end{array}$$

Here the vectors $\vec{x}$ and $\vec{y}$ have length n when both are present, and otherwise $\vec{x}$ has length $2n$. In the HT and OO cases there is a single extra parameter taken from the set $\{+,-\}$; if it is $-$ then the spectral parameters in the upper half of the grid are negated. (Note that the index n is not defined in the same way as for the enumerators such as $A_{\mathrm{HT}}(2n)$.)

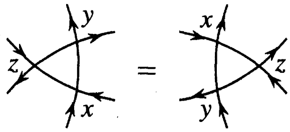
The key property of the R-matrix is that it satisfies the Yang-Baxter equation:

LEMMA 6 (Yang-Baxter equation). *If $xyz = \bar{a}$, then*



As usual the Yang-Baxter equation appears to be a massive coincidence. In our previous review of the Yang-Baxter equation [22], the R-matrix was normalized to have a particular symmetry: It was the matrix of an invariant tensor over the 2-dimensional representation of the quantum group $U_q(\mathfrak{sl}(2))$, with q related to our present parameter a. This symmetry reduced the coincidence in the equation to a single numerical equality. The spectral parameters were chosen to satisfy the equality. Here we normalize the R-matrix to reveal combinatorial symmetry rather than symmetry from quantum algebra.

Proof. Taken literally, the equation consists of 64 numerical equalities, because there are 64 ways to orient the six boundary edges on each side. However, both sides are zero unless three edges point in and three point out. This leaves 20 nonzero equations. The equation also has three kinds of symmetry: The right side is the left side rotated by 180 degrees, all arrows may be reversed, and both sides may be rotated by 120 degrees if the variables x, y, and z are cyclically permuted. By the three symmetries, 8 of the nonzero equations are tautological, and the other 12 are all equivalent. One of the 12 nontrivial equations is



In algebraic form, the equation is

$$\sigma(a\bar{y})\sigma(a^2)\sigma(a\bar{x}) = \sigma(az)\sigma(a^2)^2 + \sigma(ax)\sigma(ay)\sigma(a^2).$$

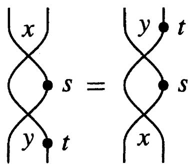
Cancelling a factor of $\sigma(a^2)$, expanding, and cancelling terms yields

$$a^2\bar{x}\bar{y} + \bar{a}^2 xy = a^3 z - z\bar{a} - \bar{z}a + \bar{a}^3\bar{z} + a^2 xy + \bar{a}^2\bar{x}\bar{y},$$

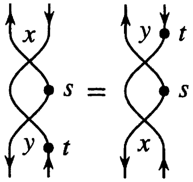
which is implied by the condition $xyz = \bar{a}$. □

We will need the reflection equation [3], [33], [7], an analogue of the Yang-Baxter equation that relates a K-matrix to the R-matrix.

LEMMA 7 (Reflection equation). *If $st = ay$ and $s\bar{t} = ax$, then*



Proof. The argument is similar to that for Lemma 6. Both sides are zero unless two boundary edges point in and two point out. There is a symmetry exchanging the two sides given by reflecting through a horizontal line and simultaneously reversing all arrows. (Note that the weights of a U-turn are not invariant under reflection alone.) Under this symmetry four of the six nonzero equations are tautological, and the other two are equivalent. One of these is:



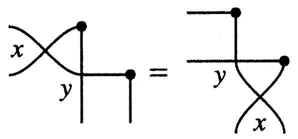
Algebraically, the equation reads:

$$\sigma(bt)\sigma(a^2)(\sigma(ay)\sigma(b\bar{s}) + \sigma(ax)\sigma(bs))$$
$$= \sigma(b\bar{t})\sigma(a^2)(\sigma(ax)\sigma(b\bar{s}) + \sigma(ay)\sigma(bs)).$$

All terms of the equation match or cancel when $st = ay$ and $s\bar{t} = ax$. □

The corner K-matrix also satisfies the reflection equation [7].

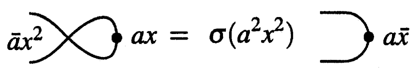
LEMMA 8. *For any x and y,*



Proof. Diagonal reflection exchanges the two sides. Both sides are zero if an odd number of boundary edges point inward. If two boundary edges point in and the other two point out, then arrow reversal is also a symmetry, because one corner must have inward arrows and the other outward arrows. These facts together imply that all cases of the equation are null or tautological. □

Finally we will need an equation that, loosely speaking, inverts a U-turn:

LEMMA 9 (Fish equation). *For any a and x,*



The proof is elementary.

4. Determinants

In this section we will establish determinant and Pfaffian formulas for the partition functions defined in Sections 2 and 3. Recall that the Pfaffian of an antisymmetric $2n \times 2n$ matrix A is defined as

$$\operatorname{Pf} A \stackrel{\text{def}}{=} \sum_{\pi \in X} (-1)^{\pi} \prod_i A_{\pi(2i-1),\pi(2i)},$$

where $X \subset S_{2n}$ has one representative in each coset of the wreath product $S_2 \wr S_n$. (Thus X admits a bijection with the set of perfect matchings of $\{1, \ldots, 2n\}$.) Recall also that

$$\det A = (\operatorname{Pf} A)^2.$$

THEOREM 10. *Let*

$$
\begin{aligned}
M(n;\vec{x},\vec{y})_{i,j} &= \frac{1}{\alpha(x_i\bar{y}_j)} \\
M^{\pm}_{\mathrm{HT}}(n;\vec{x},\vec{y})_{i,j} &= \frac{1}{\sigma(a\bar{x}_iy_j)} \pm \frac{1}{\sigma(ax_i\bar{y}_j)} \\
M_{\mathrm{U}}(n;\vec{x},\vec{y})_{i,j} &= \frac{1}{\alpha(x_i\bar{y}_j)} - \frac{1}{\alpha(x_iy_j)} \\
M_{\mathrm{UU}}(n;\vec{x},\vec{y})_{i,j} &= \frac{\sigma(b\bar{y}_j)\sigma(cx_i)}{\sigma(ax_i\bar{y}_j)} - \frac{\sigma(b\bar{y}_j)\sigma(c\bar{x}_i)}{\sigma(a\bar{x}_i\bar{y}_j)} \\
&\quad - \frac{\sigma(by_j)\sigma(cx_i)}{\sigma(ax_iy_j)} + \frac{\sigma(by_j)\sigma(c\bar{x}_i)}{\sigma(a\bar{x}_iy_j)} \\
M^{(k)}_{\mathrm{QT}}(n;\vec{x})_{i,j} &= \frac{\sigma(\bar{x}_i^k x_j^k)}{\alpha(\bar{x}_ix_j)} \\
M_{\mathrm{O}}(n;\vec{x})_{i,j} &= \frac{\sigma(\bar{x}_ix_j)}{\alpha(x_ix_j)} \\
M_{\mathrm{OO}}(n;\vec{x})_{i,j} &= \sigma(\bar{x}_ix_j)\left(\frac{c^2}{\sigma(ax_ix_j)} + \frac{b^2}{\sigma(a\bar{x}_i\bar{x}_j)}\right) \\
M^{(1)}_{\mathrm{UO}}(n;\vec{x})_{i,j} &= \sigma(\bar{x}_ix_j)\sigma(x_ix_j)\left(\frac{1}{\alpha(x_ix_j)} - \frac{1}{\alpha(\bar{x}_ix_j)}\right) \\
M^{(2)}_{\mathrm{UO}}(n;\vec{x})_{i,j} &= \sigma(\bar{x}_ix_j)\sigma(x_ix_j)\left(\frac{\sigma(cx_i)\sigma(cx_j)}{\sigma(ax_ix_j)} - \frac{\sigma(cx_i)\sigma(c\bar{x}_j)}{\sigma(ax_i\bar{x}_j)}\right. \\
&\quad \left. - \frac{\sigma(c\bar{x}_i)\sigma(cx_j)}{\sigma(a\bar{x}_ix_j)} + \frac{\sigma(c\bar{x}_i)\sigma(c\bar{x}_j)}{\sigma(a\bar{x}_i\bar{x}_j)}\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
Z(n;\vec{x},\vec{y}) &= \frac{\sigma(a^2)^n\prod_{i,j}\alpha(x_i\bar{y}_j)}{\prod_{i<j}\sigma(\bar{x}_ix_j)\sigma(y_i\bar{y}_j)}(\det M) \\
Z^{\pm}_{\mathrm{HT}}(n;\vec{x},\vec{y}) &= \frac{\sigma(a^2)^n\prod_{i,j}\alpha(x_i\bar{y}_j)^2}{\prod_{i<j}\sigma(\bar{x}_ix_j)^2\sigma(y_i\bar{y}_j)^2}(\det M)(\det M^{\pm}_{\mathrm{HT}}) \\
Z_{\mathrm{U}}(n;\vec{x},\vec{y}) &= \frac{\sigma(a^2)^n\prod_i\sigma(b\bar{y}_i)\sigma(a^2x_i^2)\prod_{i,j}\alpha(x_i\bar{y}_j)\alpha(x_iy_j)}{\prod_{i<j}\sigma(\bar{x}_ix_j)\sigma(y_i\bar{y}_j)\prod_{i\leq j}\sigma(\bar{x}_i\bar{x}_j)\sigma(y_iy_j)} \\
&\quad \cdot(\det M_{\mathrm{U}}) \\
Z_{\mathrm{UU}}(n;\vec{x},\vec{y}) &= \frac{\sigma(a^2)^n\prod_i\sigma(a^2x_i^2)\sigma(a^2\bar{y}_i^2)\prod_{i,j}\alpha(x_i\bar{y}_j)^2\alpha(x_iy_j)^2}{\prod_{i<j}\sigma(\bar{x}_ix_j)^2\sigma(y_i\bar{y}_j)^2\prod_{i\leq j}\sigma(\bar{x}_i\bar{x}_j)^2\sigma(y_iy_j)^2} \\
&\quad \cdot(\det M_{\mathrm{U}})(\det M_{\mathrm{UU}})
\end{aligned}
$$

$$
\begin{aligned}
Z_{\mathrm{QT}}(n;\vec{x}) &= \frac{\sigma(a^2)^n\sigma(a)^{3n}\prod_{i<j\leq 2n}\alpha(\bar{x}_i x_j)^2}{\prod_{i<j\leq 2n}\sigma(\bar{x}_i x_j)^2} \\
&\quad\cdot(\mathrm{Pf}\, M_{\mathrm{QT}}^{(1)})(\mathrm{Pf}\, M_{\mathrm{QT}}^{(2)}) \\
Z_{\mathrm{O}}(n;\vec{x}) &= \frac{\sigma(a^2)^n\prod_{i<j\leq 2n}\alpha(x_i x_j)}{\prod_{i<j\leq 2n}\sigma(\bar{x}_i x_j)}(\mathrm{Pf}\, M_{\mathrm{O}}) \\
Z_{\mathrm{OO}}(n;\vec{x}) &= \frac{\sigma(a^2)^n\prod_{i<j\leq 2n}\alpha(x_i x_j)^2}{\prod_{i<j\leq 2n}\sigma(\bar{x}_i x_j)^2}(\mathrm{Pf}\, M_{\mathrm{O}})(\mathrm{Pf}\, M_{\mathrm{OO}}) \\
Z_{\mathrm{UO}}(n;\vec{x}) &= \frac{b^{2n}\sigma(a^2)^n\sigma(a)^{2n}\prod_{i\leq 2n}\sigma(a^2x_i^2)}{\prod_{i<j\leq 2n}\sigma(\bar{x}_i x_j)^2\prod_{i\leq j\leq 2n}\sigma(x_i x_j)^2} \\
&\quad\cdot\prod_{i<j\leq 2n}\alpha(\bar{x}_i x_j)^2\alpha(x_i x_j)^2(\mathrm{Pf}\, M_{\mathrm{UO}}^{(1)})(\mathrm{Pf}\, M_{\mathrm{UO}}^{(2)}).
\end{aligned}
$$

We call the first four partition functions the *determinant partition functions* and the other four the *Pfaffian partition functions.*

Remark. The partition function $Z_{\mathrm{U}}(n;\vec{x},\vec{y})$, the Tsuchiya determinant, is nearly invariant if $\vec{x}$ is exchanged with $\vec{y}$. Similarly $Z_{\mathrm{O}}(n;\vec{x})$ is nearly invariant if each x_i is replaced with $\bar{x}_i$. We have no direct explanation for these symmetries. Note that the first symmetry is less apparent in Tsuchiya's matrix M [37, eq. (42)], which has an asymmetric factor

$$F_{ij} = \frac{\sinh(\zeta_- + \lambda_j)}{\sinh(\lambda_j + \omega_i)} + \frac{\sinh(\zeta_- - \lambda_j)}{\sinh(\lambda_j - \omega_i)}.$$

In this expression ω_i, λ_j, and ζ_- are obtained from x_i, y_j, and b by reparametrization. If we factor this expression,

$$F_{ij} = \frac{\sinh(2\lambda_j)\sinh(\zeta_- - \omega_i)}{\sinh(\lambda_j + \omega_i)\sinh(\lambda_j - \omega_i)},$$

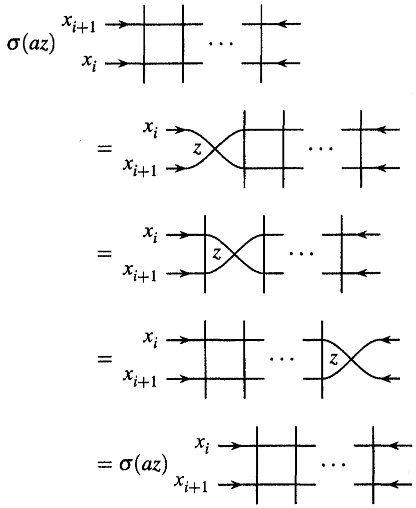
we can then pull the asymmetric factors out of the determinant since they each depend on only one of the two indices i and j. This also explains why the K-matrix parameter ζ_- or b need not appear in the matrix M_U.

The proof of Theorem 10 uses recurrence relations that determine both sides. The relations are expressed in Lemmas 11, 12, 13, and 14. Indeed, the first three of these lemmas are obvious for the right-hand sides of Theorem 10; only Lemma 14 needs to be argued for both sides.

LEMMA 11 (Baxter, Sklyanin). *Each of the partition functions in Theorem 10 is symmetric in the coordinates of $\vec{x}$. Each determinant partition function is symmetric in the coordinates of $\vec{y}$. The partition functions $Z_{\mathrm{U}}(n;\vec{x},\vec{y})$*

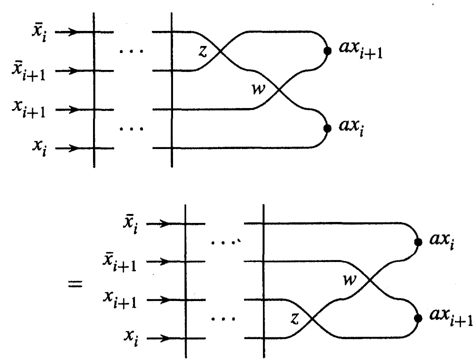
and $Z_{\mathrm{UU}}(n;\vec{x},\vec{y})$ *gain a factor of* $\sigma(a^2\bar{x}_i^2)/\sigma(a^2x_i^2)$ *if* x_i *is replaced by* $\bar{x}_i$ *for a single* i*. Similarly* $Z_{\mathrm{UU}}(n;\vec{x},\vec{y})$ *gains* $\sigma(a^2y_i^2)/\sigma(a^2\bar{y}_i^2)$ *under* $y_i \mapsto \bar{y}_i$ *and* $Z_{\mathrm{UO}}(n;\vec{x})$ *gains* $\sigma(a^2\bar{x}_i^2)/\sigma(a^2x_i^2)$ *under* $x_i \mapsto \bar{x}_i$*.*

Proof. Invariance of $Z(n;\vec{x},\vec{y})$ is an illustrative case. We exchange x_i with x_{i+1} for any $i \le n-1$ by crossing the corresponding lines at the left side. If the spectral parameter of the crossing is $z = \bar{a}x_i\bar{x}_{i+1}$, we can move it to the right side using the Yang-Baxter equation (Lemma 6) and then remove it:



The argument for symmetry in $\vec{x}$ is exactly the same for all of the square ice grids without U-turns. If the grid has diagonal boundary with corner vertices, we can bounce the crossing off of it using Lemma 8.

If the grid has U-turn boundary on the right, we exchange x_i with x_{i+1} by crossing the $\bar{x}_i$ line over the two lines above it. We let the spectral parameters of these two crossings be $z = \bar{a}\bar{x}_ix_{i+1}$ and $w = \bar{a}\bar{x}_i\bar{x}_{i+1}$. We move both crossings to the right using the Yang-Baxter equation, then we bounce them off of the U-turns using the reflection equation (Lemma 7):



Also if the grid has a U-turn on the right, we establish covariance under $x_i \mapsto \bar{x}_i$ by switching the lines with these two labels and eating the crossing using the fish equation (Lemma 9).

The same arguments establish symmetry in the coordinates of $\vec{y}$. All of the arguments used in combination establish the claimed properties of $Z_{\mathrm{UO}}(n;\vec{x})$. □

LEMMA 12. *The partition function $Z_{\mathrm{HT}}(n;\vec{x},\vec{y})$ gains a factor of $(\pm 1)^n$ if x_i and y_i are replaced by $\bar{x}_i$ and $\bar{y}_i$ for all i simultaneously. Similarly $Z_{\mathrm{OO},\pm}(n;\vec{x})$ is invariant under $x_i \mapsto \bar{x}_i$ and $b \mapsto c \mapsto b$.*

Proof. In both cases, the symmetry is effected by reflecting the square ice grid or the alternating-sign matrices through a horizontal line. □

For a vector $\vec{x} = (x_1, \ldots, x_n)$, let $\vec{x}' = (x_2, \ldots, x_n)$.

LEMMA 13. *If $x_1 = ay_1$, then*

$$\frac{Z(n;\vec{x},\vec{y})}{Z(n-1;\vec{x}',\vec{y}')} = \sigma(a^2)\prod_{2\le i}\sigma(a\bar{x}_i y_1)\sigma(a\bar{x}_1 y_i)$$

$$\frac{Z^{\pm}_{\mathrm{HT}}(n;\vec{x},\vec{y})}{Z^{\pm}_{\mathrm{HT}}(n-1;\vec{x}',\vec{y}')} = \pm\sigma(a^2)^2\prod_{2\le i}\sigma(a\bar{x}_i y_1)^2\sigma(a\bar{x}_1 y_i)^2$$

$$\frac{Z_{\mathrm{U}}(n;\vec{x},\vec{y})}{Z_{\mathrm{U}}(n-1;\vec{x}',\vec{y}')} = \sigma(a^2)\sigma(a^2x_1^2)\sigma(b\bar{y}_1)$$

$$[-2ex] \qquad \cdot\prod_{2\le i}\sigma(a\bar{x}_i y_1)\sigma(a\bar{x}_1 y_i)\sigma(ax_i y_1)\sigma(a\bar{x}_1\bar{y}_i)$$

$$\frac{Z_{\mathrm{UU}}(n;\vec{x},\vec{y})}{Z_{\mathrm{UU}}(n-1;\vec{x}',\vec{y}')} = \sigma(a^2)^2\sigma(a^2x_1^2)\sigma(a^2\bar{y}_1^2)\sigma(b\bar{y}_1)\sigma(cx_1)$$

$$[-2ex] \qquad \cdot\prod_{2\le i}\sigma(a\bar{x}_i y_1)^2\sigma(a\bar{x}_1 y_i)^2$$

$$\cdot\prod_{2\le i}\sigma(ax_i y_1)^2\sigma(a\bar{x}_1\bar{y}_i)^2.$$

If $x_2 = ax_1$, then

$$\frac{Z_{\mathrm{QT}}(n;\vec{x})}{Z_{\mathrm{QT}}(n-1;\vec{x}'')} = \sigma(a)^2\sigma(a^2)^2\prod_{3\le i\le 2n}\sigma(ax_i\bar{x}_1)^2\sigma(a\bar{x}_i x_2)^2.$$

If $x_2 = \bar{a}\bar{x}_1$, then

$$\frac{Z_{\mathrm{O}}(n;\vec{x})}{Z_{\mathrm{O}}(n-1;\vec{x}'')} = \sigma(a^2)\prod_{3\le i\le 2n}\sigma(a\bar{x}_1\bar{x}_i)\sigma(a\bar{x}_2\bar{x}_i)$$

$$\frac{Z_{\mathrm{OO}}(n;\vec{x})}{Z_{\mathrm{OO}}(n-1;\vec{x}'')} = c^2\sigma(a^2)^2 \prod_{3\le i\le 2n} \sigma(a\bar{x}_1\bar{x}_i)^2\sigma(a\bar{x}_2\bar{x}_i)^2$$

$$\frac{Z_{\mathrm{UO}}(n;\vec{x})}{Z_{\mathrm{UO}}(n-1;\vec{x}'')} = b^2\sigma(a)^2\sigma(a^2)^2\sigma(a^2x_1^2)\sigma(a^2x_2^2)$$

$$\cdot\sigma(cx_1)\sigma(cx_2) \prod_{3\le i\le 2n} \sigma(a\bar{x}_1\bar{x}_i)^2\sigma(a\bar{x}_2\bar{x}_i)^2$$

$$\cdot \prod_{3\le i\le 2n} \sigma(a\bar{x}_1 x_i)^2\sigma(a\bar{x}_2 x_i)^2.$$

Proof. This lemma is clearer in the alternating-sign matrix model than it is in the square ice model. The partition function $Z(n;\vec{x},\vec{y})$ is a sum over $n\times n$ alternating-sign matrices in which each entry of the matrix has a multiplicative weight. When $y_1 = ax_1$, the weight of a 0 in the southwest corner is 0. Consequently this corner is forced to be 1 and the left column and bottom row are forced to be 0, as in Figure 16. The sum reduces to one over $(n-1)\times(n-1)$ ASMs. The only discrepancy between $Z(n;\vec{x},\vec{y})|_{y_1=ax_1}$ and $Z(n-1;\vec{x}',\vec{y}')$ is the weights of the forced entries, which the lemma lists as factors.

$$\left(\begin{array}{c|cccc} 0 & 0 & 0 & + & 0 \\ 0 & + & 0 & - & + \\ 0 & 0 & 0 & + & 0 \\ 0 & 0 & + & 0 & 0 \\ \hline + & 0 & 0 & 0 & 0 \end{array}\right)$$

Figure 16. ASM entries forced by $y_1 = ax_1$.

The argument in the other determinant cases is identical. The argument in the Pfaffian cases is only slightly different: All QTSASMs have zeroes in the corners, and the specialization $x_2 = ax_1$ instead forces a 1 next to each corner and zeroes the first two rows and columns from each edge. Likewise the specialization $x_2 = \bar{a}\bar{x}_1$ forces a 1 next to each corner of an OSASM or an OOSASM and a 1 in the third row entry bottom of a UOSASM, and several rows and columns of zeroes in each of these cases. □

Define the *width* of a Laurent polynomial to be the difference in degree between the leading and trailing terms. (For example, $q^3 - q^{-2}$ has width 5.)

LEMMA 14. *Both sides of each equation of Theorem* 10 *are Laurent polynomials in each coordinate of $\vec{x}$ (and $\vec{y}$ in the determinant cases) and their widths in x_1 (y_1 in the determinant cases) are as given in* Table 1.

To conclude the proof of Theorem 10, we claim that Lemmas 11, 12, and 13 inductively determine both sides by Lagrange interpolation. (To begin the induction each partition function is set to 1 when $n = 0$.) If a Laurent polynomial of width w has prespecified leading and trailing exponents, it is determined by $w+1$ of its values. Each of our partition functions is a centered Laurent polynomial in x_1 (in the Pfaffian cases) or y_1 (in the determinant cases). Moreover each is either an even function or an odd function. Thus we only need $w+1$ specializations, where w is the width in x_1^2 (or y_1^2).

These widths are summarized in Table 1. To compute them, observe that each 0 entry in the bottom row of an ASM contributes 1 to the width. In the UASM and UUASM cases, it is the bottom two rows, and the U-turn itself contributes 1 to the width as well. In the QTSASM case, the corner entries always have weight $\sigma(a)$ and do not contribute to the width. Lemmas 11 and 13 together provide many specializations which are listed in Table 1. Note that Lemma 11 implies that $\sigma(a^2x_1^2)$ divides $Z_{\mathrm{UU}}(n;\vec{x},\vec{y})\ Z_{\mathrm{UO}}(n;\vec{x},vy)$, which provides an extra specialization in these two cases. In conclusion, it is easy to check that there are enough specializations to match the widths.

Function	Width	Specializations
$Z(n;\vec{x},\vec{y})$	$n-1$	$y_1 = ax_i$
$Z^{\pm}_{\mathrm{HT}}(n;\vec{x},\vec{y})$	$2n-1$	$y_1 = a^{\pm 1}x_i$
$Z_{\mathrm{U}}(n;\vec{x},\vec{y})$	$2n-1$	$y_1 = ax_i^{\pm 1}$
$Z_{\mathrm{UU}}(n;\vec{x},\vec{y})$	$4n$	$y_1 = a^{\pm 1}x_i^{\pm 1}$, a
$Z_{\mathrm{QT}}(n;\vec{x})$	$4n-3$	$x_1 = a^{\pm 1}x_i$
$Z_{\mathrm{O}}(n;\vec{x})$	$2n-2$	$x_1 = a\bar{x}_i$
$Z_{\mathrm{OO}}(n;\vec{x})$	$4n-3$	$x_1 = a^{\pm 1}\bar{x}_i$
$Z_{\mathrm{UO}}(n;\vec{x})$	$8n-4$	$x_1 = a^{\pm 1}x_i^{\pm 1}$, $\bar{a}$

Table 1. Widths and specializations of partition functions.

Remark. The formulas in Theorem 10 are even more special than Lemmas 11 through 14 suggest. Among the evidence for this, the recurrence relations still hold with only slight modifications if all spectral parameters in the QT, UU, and UO grids are multiplied by an extra parameter z. Similarly the spectral parameters in the top halves of the HT and OO grids may be multiplied by an arbitrary z instead of by ± 1. However, we were not able to generalize Theorem 10 to include this parameter.

Lemma 13 reveals another subtlety, namely that

$$\frac{Z^{+}_{\mathrm{HT}}(n;\vec{x},\vec{y})}{Z^{+}_{\mathrm{HT}}(n-1;\vec{x}',\vec{y}')} = \left(\frac{Z(n;\vec{x},\vec{y})}{Z(n-1;\vec{x}',\vec{y}')}\right)^2$$

at every specialization $y_i = a^{\pm 1} x_j$. Since this coincidence holds for enough specializations to determine $Z^+_{\mathrm{HT}}(n;\vec{x},\vec{y})$ entirely, one might suppose that

$$Z^+_{\mathrm{HT}}(n;\vec{x},\vec{y}) = Z(n;\vec{x},\vec{y})^2.$$

But then $Z^+_{\mathrm{HT}}(n;\vec{x},\vec{y})$ would be an even function of y_1, while in reality it is an odd function. The other symmetry classes involving half-turn rotation have similar behavior.

5. Factor exhaustion

In this section we derive several round determinants and Pfaffians depending on two and three parameters. We will later identify special cases of the determinants and Pfaffians with those appearing in Theorem 10, and they will specialize further to establish the enumerations in Theorems 2, 3, and 5.

Since the formulas in this section may seem complicated, we recommend verifying that they are round without worrying about their exact form in the first reading. For this purpose we give a more precise definition of roundness that also applies to polynomials. A term R_n in a sequence of rational polynomials depending on one or more variables is *round* if it is a ratio of products of constants, monomials, and differences of two monic terms. All exponents and constant factors should grow polynomially in n or be independent of n. For example, $n!3^n(q+p^n)$ is round and Gaussian binomial coefficients are round. Note that a round polynomial in a single variable must be a product of cyclotomic polynomials, which is part of the motivation for the term "round". Roundness is preserved when a variable is set to 1 or to a product of other variables. In a later reading one can verify the explicit formulas. This is a tedious but elementary computation, because all round expressions involved have an explicit and regular form. As a warmup the reader can verify that the expressions for $A(n)$ in Theorems 1 and 2 coincide.

We begin with the classic Cauchy double alternant and a Pfaffian generalization found independently by Stembridge and by Laksov, Lascoux, and Thorup [16], [35], [23], [36].

THEOREM 15 (Cauchy, S. L. L. T.). *Let*

$$\begin{aligned} C_1(\vec{x},\vec{y})_{i,j} &= \frac{1}{x_i+y_j} \\ C_2(\vec{x},\vec{y})_{i,j} &= \frac{1}{x_i+y_j} - \frac{1}{1+x_iy_j}. \end{aligned}$$

For $1 \le i,j \le 2n$, *let*

$$\begin{aligned} C_3(\vec{x})_{i,j} &= \frac{x_j-x_i}{x_i+x_j} \\ C_4(\vec{x})_{i,j} &= \frac{x_j-x_i}{1-x_ix_j}. \end{aligned}$$

Then

$$\begin{aligned}
\det C_1 &= \frac{\prod_{i<j}(x_j - x_i)(y_j - y_i)}{\prod_{i,j}(x_i + y_j)} \\
\det C_2 &= \frac{\prod_{i<j}(1 - x_i x_j)(1 - y_i y_j)(x_j - x_i)(y_j - y_i)}{\prod_{i,j}(x_i + y_j)(1 + x_i y_j)} \\
&\quad \cdot \prod_i (1 + x_i)(1 + y_i) \\
\operatorname{Pf} C_3 &= \prod_{i<j\le 2n} \frac{x_i - x_j}{x_i + x_j} \\
\operatorname{Pf} C_4 &= \prod_{i<j\le 2n} \frac{x_i - x_j}{1 - x_i x_j}.
\end{aligned}$$

Proof. Our proof is by the factor exhaustion method [20]. The determinant $\det C_1$ is divisible by $x_j - x_i$ because when $x_i = x_j$, two rows of C_1 are proportional. Likewise it is also divisible by $y_j - y_i$. At the same time, the polynomial

$$\prod_{i,j}(x_i + y_j)(\det C_1)$$

has degree $n^2 - n$, so it has no room for other nonconstant factors. This determines $\det C_1$ up to a constant, which can be found inductively by setting $x_1 = -y_1$.

The determinant $\det C_2$ is argued the same way. The Pfaffians $\operatorname{Pf} C_3$ and $\operatorname{Pf} C_4$ are also argued the same way; here the constant factor can be found by setting $x_1 = \bar{x}_2$. □

Next we evaluate four determinants in the variables p and q. We use two more functions similar to σ and α from Section 4:

$$\gamma(q) = q^{1/2} - q^{-1/2} \qquad \tau(q) = q^{1/2} + q^{-1/2}.$$

THEOREM 16. *Let*

$$\begin{aligned}
T_1(p,q)_{i,j} &= \frac{\gamma(q^{n+j-i})}{\gamma(p^{n+j-i})} \\
T_2(p,q)_{i,j} &= \frac{\tau(q^{j-i})}{\tau(p^{j-i})} \\
T_3(p,q)_{i,j} &= \frac{\gamma(q^{n+j+i})}{\gamma(p^{n+j+i})} - \frac{\gamma(q^{n+j-i})}{\gamma(p^{n+j-i})} \\
T_4(p,q)_{i,j} &= \frac{\tau(q^{j+i})}{\tau(p^{j+i})} - \frac{\tau(q^{j-i})}{\tau(p^{j-i})}.
\end{aligned}$$

Then

$$\begin{aligned}
\det T_1 &= \frac{\prod_{i\neq j}\gamma(p^{j-i})\prod_{i,j}\gamma(qp^{j-i})}{\prod_{i,j}\gamma(p^{n+j-i})}\\
\det T_2 &= (-1)^{\binom{n}{2}}\frac{2^n\prod_{\substack{i\neq j\\ 2|j-i}}\gamma(p^{j-i})^2\prod_{\substack{i,j\\ 2\nmid j-i}}\gamma(qp^{j-i})}{\prod_{i,j}\tau(p^{j-i})}\\
\det T_3 &= \frac{\prod_{i<j\le 2n}\gamma(p^{j-i})\prod_{\substack{i,j\le 2n+1\\ 2|j}}\gamma(qp^{j-i})}{\prod_{i,j}\gamma(p^{n+j-i})\gamma(p^{n+j+i})}\\
\det T_4 &= \frac{2^n\prod_{i<j\le n}\gamma(p^{2(j-i)})^2\prod_{\substack{i,j\le 2n+1\\ 2\nmid i,2|j}}\gamma(qp^{j-i})}{\prod_{i,j}\tau(p^{j-i})\tau(p^{j+i})}.
\end{aligned}$$

Proof. Factor exhaustion. We first view each determinant as a fractional Laurent polynomial in q. By choosing special values of q, we will find enough factors in each determinant to account for their entire width, thus determining them up to a rational factor $R(p)$. (Each determinant is a centered Laurent polynomial in q with fractional exponents. The notion of width make sense for these.) We will derive this factor by a separate method.

For example, if $0 \le k < n$, then $\det T_1$ is divisible by $\gamma(qp^{-k})^{n-k}$ because

$$T_1(p,p^k)_{i,j} = \sum_{\frac{1-k}{2}\le \ell \le \frac{k-1}{2}} p^{\ell(n+j-i)}.$$

Evidently $T_1(p,p^k)$ is a sum of k rank 1 matrices at this specialization, so its determinant has an $(n-k)$-fold root at $q = p^k$. Likewise $T(p,p^{-k})$ also has rank k and $\gamma(qp^k)^{n-k}$ also divides $\det T_1$. All four of the determinants have this behavior. In each case, the singular values of q can be read from the product formulas for the determinants. The only detail that changes is the form of each rank 1 term, which is summarized in Table 2.

Matrix	Rank 1 terms	Extra q value
T_1	$p^{-\ell i}p^{\ell(n+j)}$	∞
T_2	$p^{-\ell i}p^{\ell j}$	1
T_3	$(p^{-\ell i}-p^{\ell i})(p^{\ell(n+j)}-p^{-\ell(n+j)})$	∞
T_4	$(p^{-\ell i}-p^{\ell i})(p^{\ell j}-p^{-\ell j})$	∞

Table 2. Details of factor exhaustion for Theorem 16.

Finally the q-independent factor $R(p)$ can be found by examining the coefficient of the leading power of q, or equivalently, taking the limit $q \to \infty$. For example

$$\frac{T_1(p,q)_{i,j}}{q^{(n+j-i)/2}} \to \frac{1}{\gamma(p^{n+j-i})} = p^{(n+i+j)/2}C(\vec{x},\vec{y})$$

as $q \to \infty$ with

$$x_i = p^{-i} \qquad y_j = p^{n+j}.$$

In this case $R(p)$ is given by $\det C_1$ in Theorem 15. This happens in each case, although for the matrix T_2 it is slightly more convenient to specialize to $q = 1$. The best extra value of q in all four cases is given in Table 2. □

Finally we evaluate two three-variable Pfaffians which are like the determinants in Theorem 16.

THEOREM 17. *For $i, j \le 2n$, let*

$$\begin{aligned} T_5(p,q,r)_{i,j} &= \frac{\gamma(q^{j-i})\gamma(r^{j-i})}{\gamma(p^{j-i})} \\ T_6(p,q,r)_{i,j} &= \gamma(p^{j+i})\gamma(p^{j-i})\left(\frac{\gamma(q^{j+i})}{\gamma(p^{j+i})} - \frac{\gamma(q^{j-i})}{\gamma(p^{j-i})}\right) \\ &\quad \left(\frac{\gamma(r^{j+i})}{\gamma(p^{j+i})} - \frac{\gamma(r^{j-i})}{\gamma(p^{j-i})}\right). \end{aligned}$$

when $i \neq j$ and

$$\begin{aligned} T_5(p,q,r)_{i,i} &= 0 \\ T_6(p,q,r)_{i,i} &= 0. \end{aligned}$$

Then

$$\begin{aligned} \operatorname{Pf} T_5 &= \frac{\prod_{i<j} \gamma(p^{j-i})^4 \prod_{i,j} \gamma(qp^{j-i})\gamma(rp^{j-i})}{\prod_{i<j\le 2n} \gamma(p^{j-i})} \\ \operatorname{Pf} T_6 &= \frac{\prod_{i<j\le 2n} \gamma(p^{j-i}) \prod_{\substack{i,j\le 2n+1 \\ 2|j}} \gamma(qp^{j-i})\gamma(rp^{j-i})}{\prod_{i<j\le 2n} \gamma(p^{j+i})}. \end{aligned}$$

Proof. Factor exhaustion in both q and r. If $0 \le k < n$, then

$$T_5(p,p^k,r)_{i,j} = \sum_{\frac{1-k}{2} \le \ell \le \frac{k-1}{2}} r^{1/2} p^{\ell(j-i)} - \sum_{\frac{1-k}{2} \le \ell \le \frac{k-1}{2}} r^{-1/2} p^{\ell(j-i)}$$

is, as written, a sum of $2k$ rank 1 matrices. Therefore the Pfaffian, whose square is the determinant, is divisible by $\gamma(qp^{-k})^{n-k}$. The same argument applies to $T_5(p,p^{-k},r)$. It also applies to $T_5(p,q,p^{\pm k})$ since T_5 is symmetric in q and r.

This determines $\operatorname{Pf} T_5$ up to a factor $R(p)$ depending only on p. This factor can be determined by taking the limit $r \to \infty$:

$$\lim_{r\to\infty} \frac{T_5(p,q,r)_{i,j}}{r^{(|i-n-\frac{1}{2}|+|j-n-\frac{1}{2}|)/2}} = \begin{cases} T_1(p,q)_{i,j-n} & i \le n < j \\ -T_1(p,q)_{j,i-n} & j \le n < i \\ 0 \end{cases}.$$

In other words, after rescaling rows and columns, $T_5(p,q,r)$ has a block matrix limit:

$$\lim_{r\to\infty} r^{\bullet}T_5(p,q,r) = \left(\begin{array}{c|c} 0 & T_1(p,q) \\ \hline -T_1(p,q)^T & 0 \end{array}\right).$$

(The bullet $\bullet$ is the exponent above that is different for different rows and columns.) This establishes that the leading coefficient of $\operatorname{Pf} T_5(p,q,r)$ as a polynomial in r is

$$(-1)^{\binom{n}{2}} \det T_1(p,q),$$

which in turn determines $R(p)$.

The Pfaffian $\operatorname{Pf} T_6$ is argued the same way. To find the factor $R(p)$ which is independent of q and r, we take the limit $r,q \to \infty$. In this limit $\operatorname{Pf} T_6$ reduces to a special case of $\operatorname{Pf} C_4$ in Theorem 15. $\square$

Remark. Several other specializations of the determinants in Theorem 16 and the Pfaffian in Theorem 17 are special cases of Theorem 15 and other determinants and Pfaffians such as these [20]:

$$\det\left\{x_i^{j-1}\right\} \qquad \det\left\{\gamma(x_i^{2j-1})\right\}.$$

For example, $T_1(q^2,q)$ is also a Cauchy double alternant, while $T_1(p,p^n)$ is the product of two (rescaled) Vandermonde matrices. Any of these intersections may be used to determine the q-independent factor in the factor exhaustion method. There are also other round determinants like the ones in Theorem 16 which we do not need, for example

$$\det\left\{\frac{\gamma(q^{n+j-i})}{\gamma(p^{n+j-i})} - \frac{\gamma(q^{n+j+i-1})}{\gamma(p^{n+j+i-1})}\right\}.$$

These examples suggest the following more general problem: Let M be an $n\times n$ matrix such that $M_{i,j}$ is a rational polynomial in a fixed number of variables, such as p, q, and r, and in exponentials of them such as p^i, q^j, and r^n. When is $\det M$ round? What if M_{ij} is a rational polynomial in variables such as x_i and y_j?

6. Enumerations and divisibilities

In this section we relate the quantities appearing in the other sections to prove the results in Section 1.

Let

$$\begin{aligned} \vec{1} &= (1,1,1,1,\ldots,1) \\ x &= a^2+2+\bar{a}^2 \\ y &= \sigma(ba)/\sigma(b\bar{a}) \\ z &= \sigma(ca)/\sigma(c\bar{a}). \end{aligned}$$

Then most of the generating functions in Section 1 can be expressed in terms of the partition functions in Section 4,

$$\begin{aligned}
A(n;x) &= \frac{Z(n;\vec{1},\vec{1})}{\sigma(a)^{n^2-n}\sigma(a^2)^n} \\
A_{\mathrm{HT}}(2n;x,\pm 1) &= \frac{Z^{\pm}_{\mathrm{HT}}(n;\vec{1},\vec{1})}{\sigma(a)^{2n^2-n}\sigma(a^2)^n} \\
A_{\mathrm{U}}(2n;x,y) &= \frac{Z_{\mathrm{U}}(n;\vec{1},\vec{1})}{\sigma(a)^{2n^2-n}\sigma(a^2)^n\sigma(b\bar{a})^n} \\
A_{\mathrm{UU}}(4n;x,y,z) &= \frac{Z_{\mathrm{UU}}(n;\vec{1},\vec{1})}{\sigma(a)^{4n^2-n}\sigma(a^2)^n\sigma(b\bar{a})^n\sigma(c\bar{a})^n} \\
A_{\mathrm{QT}}(4n;x) &= \frac{Z_{\mathrm{QT}}(n;\vec{1})}{\sigma(a)^{4n^2-n}\sigma(a^2)^n} \\
A_{\mathrm{O}}(2n;x) &= \frac{Z_{\mathrm{O}}(n;\vec{1})}{\sigma(a)^{2n^2-2n}\sigma(a^2)^n} \\
A_{\mathrm{UO}}(8n;x,z) &= \frac{Z_{\mathrm{UO}}(n;\vec{1},\vec{1})}{\sigma(a)^{8n^2-3n}\sigma(a^2)^n\sigma(c\bar{a})^n b^{2n}},
\end{aligned}$$

by the definition of the partition functions and the correspondence between square ice and alternating-sign matrices. The generating function $A_{\mathrm{OO}}(4n;x,y)$ requires a slightly different change of parameters: if $y = b^2/c^2$, then

$$A_{\mathrm{OO}}(4n;x,y) = \frac{Z_{\mathrm{OO}}(n;\vec{1})}{\sigma(a)^{4n^2-3n}\sigma(a^2)^n c^{2n}}.$$

The generating function $A_{\mathrm{U}}(n;x,y)$ is a polynomial of degree n in y and it is easy to show that the leading and trailing coefficients count VSASMs, so we can say that

$$A_{\mathrm{V}}(2n+1;x) = A_{\mathrm{U}}(n;x,0) = A_{\mathrm{U}}(n;x,\infty),$$

where by abuse of notation, if $P(x)$ is a polynomial (or a rational function), $P(\infty)$ denotes the top-degree coefficient. Likewise $A_{\mathrm{UU}}(n;x,y,z)$ has bidegree (n,n) in y and z and the corner coefficients count VHSASMs and VHPASMs:

$$\begin{aligned}
A_{\mathrm{VHP}}(4n+2;x) &= A_{\mathrm{UU}}(n;x,0,\infty) = A_{\mathrm{UU}}(n;x,\infty,0) \\
A_{\mathrm{VH}}(4n+1;x) &= A_{\mathrm{UU}}(n;x,\infty,\infty) \\
A_{\mathrm{VH}}(4n+3;x) &= A_{\mathrm{UU}}(n;x,0,0).
\end{aligned}$$

We can reverse these relations by defining

$$\begin{aligned}
(1) \qquad Z^{\pm,(2)}_{\mathrm{HT}}(n;\vec{x},\vec{y}) &= \frac{\prod_{i,j}\alpha(x_i\bar{y}_j)(\det M^{\pm}_{\mathrm{HT}})}{\prod_{i<j}\sigma(\bar{x}_i x_j)\sigma(y_i\bar{y}_j)} \\
Z^{(2)}_{\mathrm{UU}}(n;\vec{x},\vec{y}) &= \frac{\prod_{i,j}\alpha(x_i\bar{y}_j)\alpha(x_i y_j)(\det M_{\mathrm{UU}})}{\prod_{i<j}\sigma(\bar{x}_i x_j)\sigma(y_i\bar{y}_j)\prod_{i\le j}\sigma(\bar{x}_i\bar{x}_j)\sigma(y_i y_j)}
\end{aligned}$$

$$
\begin{aligned}
Z^{(k)}_{\mathrm{QT}}(n;\vec{x}) &= \frac{\prod_{i<j\leq 2n}\alpha(\bar{x}_i x_j)(\operatorname{Pf} M^{(k)}_{\mathrm{QT}})}{\prod_{i<j\leq 2n}\sigma(\bar{x}_i x_j)} \\
Z^{(2)}_{\mathrm{OO}}(n;\vec{x}) &= \frac{\prod_{i<j\leq 2n}\alpha(x_i x_j)(\operatorname{Pf} M_{\mathrm{OO}})}{\prod_{i<j\leq 2n}\sigma(\bar{x}_i x_j)} \\
Z^{(k)}_{\mathrm{UO}}(n;\vec{x}) &= \frac{\prod_{i<j\leq 2n}\alpha(\bar{x}_i x_j)\alpha(x_i x_j)(\operatorname{Pf} M^{(k)}_{\mathrm{UO}})}{\prod_{i<j\leq 2n}\sigma(\bar{x}_i x_j)\prod_{i\leq j\leq 2n}\sigma(x_i x_j)}
\end{aligned}
$$

and

$$
\begin{aligned}
A^{(2)}_{\mathrm{HT}}(2n;x,\pm 1) &= \sigma(a)^{n-n^2} Z^{\pm,(2)}_{\mathrm{HT}}(2n;\vec{1},\vec{1}) \\
A^{(2)}_{\mathrm{UU}}(4n;x,y,z) &= \sigma(a)^{n-2n^2}\sigma(b\bar{a})^{-n}\sigma(c\bar{a})^{-n} Z^{(2)}_{\mathrm{UU}}(4n;\vec{1},\vec{1}) \\
A^{(k)}_{\mathrm{QT}}(4n;x) &= \sigma(a)^{n-2n^2} Z^{(k)}_{\mathrm{QT}}(4n;\vec{1}) \\
A^{(2)}_{\mathrm{OO}}(4n;x,y) &= c^{-2n}\sigma(a)^{2n-2n^2} Z^{(2)}_{\mathrm{OO}}(4n;\vec{1}) \\
A^{(1)}_{\mathrm{UO}}(8n;x,z) &= \sigma(a)^{n-4n^2} Z^{(1)}_{\mathrm{UO}}(8n;\vec{1},\vec{1}) \\
A^{(2)}_{\mathrm{UO}}(8n;x,z) &= \sigma(a)^{n-4n^2}\sigma(c\bar{a})^{-n} Z^{(2)}_{\mathrm{UO}}(8n;\vec{1},\vec{1}),
\end{aligned}
$$

using the same correspondence between a, b, and c with x, y, and z (which slightly differs in the case of OOSASMs). The observation that all of these quantities must be polynomials establishes the factorizations of A_{HT}, A_{OO}, A_{QT}, A_{UU}, and A_{UO} in Theorem 4.

For vectors $\vec{x}$ and $\vec{y}$, let

$$(\vec{x},\vec{y}) = (x_1,\ldots,x_n,y_1,\ldots,y_n)$$

denote their concatenation, and let exponentiation of vectors denote coordinate-wise exponentiation:

$$\vec{x}^k = (x_1^k, x_2^k, \ldots, x_n^k).$$

Then the matrices

$$
\begin{gathered}
M(2n;(\vec{x},\vec{x}^{-1}),(\vec{y},\vec{y}^{-1})) \\
M_{\mathrm{HT}}(2n;(\vec{x},\vec{x}^{-1}),(\vec{y},\vec{y}^{-1}))
\end{gathered}
$$

commute with the permutation matrix

$$\left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right),$$

where I_n is the $n\times n$ identity matrix. Similarly the matrices

$$
\begin{gathered}
M(2n+1;(\vec{x},1,\vec{x}^{-1}),(\vec{y},1,\vec{y}^{-1})) \\
M_{\mathrm{HT}}(2n+1;(\vec{x},1,\vec{x}^{-1}),(\vec{y},1,\vec{y}^{-1}))
\end{gathered}
$$

commute with

$$P = \left(\begin{array}{c|c|c} 0 & 0 & I_n \\ \hline 0 & 1 & 0 \\ \hline I_n & 0 & 0 \end{array}\right).$$

If $\vec{x}$ has length $2n$ and $b^2 = -c^2$, then

$$M_{\mathrm{OO}}(4n; (\vec{x}, \vec{x}^{-1}))$$

commutes with

$$P = \left(\begin{array}{c|c} 0 & I_{2n} \\ \hline I_{2n} & 0 \end{array}\right).$$

In each case we can decompose M, M_{HT}, and M_{OO} into blocks corresponding to the eigenspaces of P. The block with eigenvalue -1 is in the three cases equal to M_{U} and proportional to M_{UU} (with $b = c = \mathbf{i}$) and $M^{(1)}_{\mathrm{UO}}$. This results in the factorizations of $A(n;x)$, $A^{(2)}_{\mathrm{HT}}(n;x,1)$, and $A^{(2)}_{\mathrm{OO}}(4n;x,-1)$ in Theorem 4. Another factorization in Theorem 4 is that of $A^{(2)}_{\mathrm{HT}}(2n;x,-1)$. To establish this, observe that the matrix $M^{-}_{\mathrm{HT}}(2n;\vec{x},\vec{x})$ is antisymmetric, and that it is proportional to $M^{(1)}_{\mathrm{QT}}(2n;\vec{x})$. The latter matrix is employed for its Pfaffian, while the former for its determinant, which is the square of the Pfaffian.

The final case of Theorem 4 is the relation between $A_{\mathrm{V}}(2n+1;x)$ and $A_{\mathrm{U}}(2n;x)$. This relation is established by observing that b appears only in the normalization factor for $Z_{\mathrm{U}}(n;x)$ and not in the matrix $M_{\mathrm{U}}(n;x)$; the only step is to change variables from b to y.

Finally we establish the round enumerations in Theorems 2, 3, and 5. We review the argument from Reference 22 for $A(n)$, which is an illustrative case. Let $\omega_n = \exp(\pi\mathbf{i}/\mathbf{n})$, where $\mathbf{i^2 = -1}$. Equation **??** implies the following correspondence between x and a:

$$\begin{aligned} a = \omega_3 \quad &\Rightarrow \quad x = 1, \\ a = \omega_4 \quad &\Rightarrow \quad x = 2, \\ a = \omega_6 \quad &\Rightarrow \quad x = 3. \end{aligned}$$

For any of these values of x or a, we would like to evaluate the partition function $Z(n;\vec{1},\vec{1})$ to find $A(n;x)$ by equation **??**. Unfortunately the matrix $M(n;\vec{1},\vec{1})$ is singular. So instead we will find its determinant along a curve of parameters that includes $(\vec{1},\vec{1})$. More precisely, let

$$\vec{q}(k) = \left(q^{(k+1)/2}, q^{(k+2)/2}, \ldots, q^{(k+n)/2}\right)$$

and

$$\vec{q} = \vec{q}(0).$$

Then

$$\lim_{q\to 1} \vec{q}(k) = \vec{1},$$

and if we assume equation **??**,

$$M(n;\vec{q}(0),\vec{q}(k)) = \frac{1}{-q^{k+j-i}+x-2-q^{i-j-k}}.$$

If we further set $x=1$ and $k=n$, then

$$M(n;\vec{q}(0),\vec{q}(n)) = -T_1(q^3,q)$$

has a round determinant by Theorem 16. Computing $A(n;x)$ is then a routine but tedious simplification of round products. The argument for most of the other enumerations is the same, except that the curve of parameters is $(\vec{q},\vec{q}(n))$ for 1-enumeration in the determinant cases, $(\vec{q},\vec{q})$ for 2- and 3-enumeration in the determinant cases, and $\vec{q}$ in the Pfaffian cases. Each of the matrices is then proportional to some matrix T_i from Theorem 16 or 17. The determinants and Pfaffian for three of the 2-enumerations are round without specializing the parameters and instead reduce to a matrix C_i from Theorem 15. These variations of the argument are summarized in Table 3.

Enumeration	Parameters	Section 4	Section 5
$A(n;1)$	$a=\omega_3$	$M(\vec{q},\vec{q}(n))$	$T_1(q^3,q)$
$A(n;2)$	$a=\omega_4$	$M(\vec{x},\vec{y})$	$C_1(\vec{x}^2,\vec{y}^2)$
$A(n;3)$	$a=\omega_6$	$M(\vec{q},\vec{q})$	$T_2(q^3,q)$
$A_{\mathrm{HT}}(2n;1,1)$	$a=\omega_3$	$M^+_{\mathrm{HT}}(\vec{q},\vec{q}(n))$	$T_1(q^3,q^2)$
$A_{\mathrm{HT}}(2n;2,1)$	$a=\omega_4$	$M^+_{\mathrm{HT}}(\vec{q},\vec{q})$	$T_2(q^2,q)$
$A_{\mathrm{V}}(2n+1;1)$	$a=\omega_3$	$M_{\mathrm{U}}(\vec{q},\vec{q}(n))$	$T_3(q^3,q)$
$A_{\mathrm{V}}(2n+1;2)$	$a=\omega_4$	$M_{\mathrm{U}}(\vec{x},\vec{y})$	$C_2(\vec{x}^2,\vec{y}^2)$
$A_{\mathrm{V}}(2n+1;3)$	$a=\omega_6$	$M_{\mathrm{U}}(\vec{q},\vec{q})$	$T_4(q^3,q)$
$A^{(2)}_{\mathrm{UU}}(4n;1,1,1)$	$a=\omega_3, b=c=\omega_4$	$M_{\mathrm{UU}}(\vec{q},\vec{q}(n))$	$T_3(q^3,q^2)$
$A^{(2)}_{\mathrm{UU}}(4n;2,1,1)$	$a=b=c=\omega_4$	$M_{\mathrm{UU}}(\vec{q},\vec{q})$	$T_4(q^2,q)$
$A^{(2)}_{\mathrm{VHP}}(4n;1)$	$a=b=\omega_3, c=\bar{a}$	$M_{\mathrm{UU}}(\vec{q},\vec{q}(n))$	$T_3(q^3,q)$
$A^{(1)}_{\mathrm{QT}}(4n;1)$	$a=\omega_3$	$M^{(1)}_{\mathrm{QT}}(\vec{q})$	$T_5(q^3,q,q)$
$A^{(1)}_{\mathrm{QT}}(4n;2)$	$a=\omega_4$	$M^{(1)}_{\mathrm{QT}}(\vec{q})$	$T_5(q^4,q^2,q)$
$A^{(1)}_{\mathrm{QT}}(4n;3)$	$a=\omega_6$	$M^{(1)}_{\mathrm{QT}}(\vec{q})$	$T_5(q^6,q^3,q^2)$
$A^{(2)}_{\mathrm{QT}}(4n;1)$	$a=\omega_3$	$M^{(2)}_{\mathrm{QT}}(\vec{q})$	$T_5(q^3,q^2,q)$
$A^{(2)}_{\mathrm{QT}}(4n;2)$	$a=\omega_4$	$M^{(2)}_{\mathrm{QT}}(\vec{x})$	$C_3(\vec{x}^2)$
$A_{\mathrm{O}}(2n;1)$	$a=\omega_3$	$M_{\mathrm{O}}(\vec{q})$	$T_6(q^3,q,\infty)$
$A^{(1)}_{\mathrm{UO}}(8n;1)$	$a=\omega_3, c=\omega_4$	$M^{(1)}_{\mathrm{UO}}(\vec{q})$	$T_6(q^3,q,q)$
$A^{(2)}_{\mathrm{UO}}(8n;1,1)$	$a=\omega_3, c=\omega_4$	$M^{(2)}_{\mathrm{UO}}(\vec{q})$	$T_6(q^3,q^2,q)$

Table 3. Specializations of partition function determinants and Pfaffians.

7. Discussion

Even though each section of this article considers many types of alternating-sign matrices or determinants in parallel, the work of enumerating symmetry classes of ASMs is far from finished. Robbins [30] conjectures formulas for the number of VHSASMs and for the number of odd-order HTSASMs, QTSASMs, and DASASMs in addition to the enumerations that we have proven. Theorem 10 yields a determinant formula for the number of VHSASMs, obtained from the more general partition function $Z_{\mathrm{UU}}(n;\vec{x},\vec{y})$ by setting $a=\omega^2$ and $b=c=\omega^{\pm 2}$, and all other parameters to 1). In the enumeration of $4n+1\times 4n+1$ VHSASMs, where $b=c=\omega^2$, experiments indicate that

$$\det M_{\mathrm{UU}}(\vec{q}(-\frac{1}{2}),\vec{q}(n-\frac{1}{2}))$$

is round, but we cannot prove this. In the other case, $4n+3\times 4n+3$ VHSASMs, where $b=c=\omega^{-2}$, we could not even find a curve for which the determinant is round. This strange behavior of VHSASMs is one illustration that although we have put many classes of ASMs under one roof, the house is not completely in order.

For the other three classes conjecturally enumerated by Robbins, we could not even find a determinant formula. Nonetheless we conjecture:

QUESTION 18. *Can* DSASMs, DASASMs, TSASMs, *and odd-order* HTSASMs *and* QTSASMs *be x-enumerated in polynomial time?*

The polynomials listed in Table 4 appear to be generating functions of some type, but in most cases there is not even a proof that their coefficients are nonnegative. (We have more data than is shown in the table; the multivariate polynomials $A^{(2)}_{\mathrm{UU}}(4n;x,y,z)$, $A^{(2)}_{\mathrm{OO}}(4n;x,y)$, and $A^{(2)}_{\mathrm{UO}}(4n;x,y)$ also appear to be nonnegative.) Some of them are conjecturally related to cyclically symmetric plane partitions [30]. Indeed they are related to each other in strange ways. For example Theorem 5 establishes that if we take $x=1$, three of the polynomial series (A_{V}, A_{O}, and $A^{(2)}_{\mathrm{VHP}}$) become equal, as if to suggest that VSASMs can be x-enumerated in three different ways!

QUESTION 19. *Do the polynomials in Table* 4 *x-enumerate classes of alternating-sign matrices?*

OSASMs include the set of off-diagonal permutation matrices, which can be interpreted as the index set for the usual combinatorial formula for the Pfaffian. Like ASMs, their number is round. These observations, together with the known formulas due to Mills, Robbins, and Rumsey [26] motivate the following question:

Factor	$n=1$	$n=2$	$n=3$
$A_{\mathrm{V}}(2n+1;x)$	1	$x+2$	$x^3+6x^2+13x+6$
$\widetilde{A}_{\mathrm{V}}(2n;x)$	1	$x+6$	$x^3+12x^2+70x+60$
$A^{(1)}_{\mathrm{QT}}(4n;x)$	1	$x+3$	$x^3+8x^2+25x+15$
$A^{(2)}_{\mathrm{QT}}(4n;x)$	2	$6x+4$	$20x^3+60x^2+52x+8$
$A^{(2)}_{\mathrm{UU}}(4n;x,1,1)$	$x+4$	$x^3+9x^2+40x+16$	$x^6+16x^5+125x^4+629x^3+1036x^2+560x+64$
$\widetilde{A}^{(2)}_{\mathrm{UU}}(4n;x)$	1	$x+1$	x^3+4x^2+5x+1
$A^{(2)}_{\mathrm{VHP}}(4n+2;x)$	1	$2x+1$	$5x^3+12x^2+8x+1$
$A_{\mathrm{O}}(2n;x)$	1	3	$x^2+10x+15$
$A^{(2)}_{\mathrm{OO}}(4n;x,1)$	2	$2x+4$	$2x^3+20x^2+28x+8$
$A^{(1)}_{\mathrm{UO}}(8n;x)$	1	x^2+5x+3	$x^6+14x^5+82x^4+210x^3+239x^2+115x+15$
$\widetilde{A}^{(1)}_{\mathrm{UO}}(8n;x)$	2	$2x^2+20x+20$	$2x^6+42x^5+420x^4+1680x^3+2892x^2+2040x+360$
$A^{(2)}_{\mathrm{UO}}(8n;x,1)$	$x+4$	$x^4+12x^3+65x^2+104x+16$	$x^9+24x^8+275x^7+1966x^6+8215x^5+19144x^4$ $+21777x^3+10028x^2+1712x+64$

Table 4. Irreducible x-enumerations.

QUESTION 20. *Are there formulas for the Pfaffian of a matrix involving* OSASMs *that generalize the determinant formulas involving* ASMs?

Neither any of the enumerations that we establish, nor the various equinumerations that they imply, have known bijective proofs. Nor is it even known that two equinumerous types of ASMs index bases of the same vector space. For example, can one find an explicit isomorphism between the vector space of formal linear combinations of $2n \times 2n$ OSASMs and the vector space of formal linear combinations of $2n+1 \times 2n+1$ VSASMs?

Sogo found that $Z(n;\vec{1},\vec{1})$ satisfies the Toda chain (or Toda molecule) differential hierarchy [34], [17].

QUESTION 21. *If $\vec{x}$ and $\vec{y}$ are set to $\vec{1}$, do the partition functions in Theorem* 10 *and Table* 4 *satisfy natural differential hierarchies*?

Many other solutions to the Yang-Baxter equation are known [8]. The six-vertex solution corresponds to the Lie algebra $\mathfrak{sl}(2)$ together with its 2-dimensional representation; there are solutions for other simple Lie algebras and their representations.

QUESTION 22. *Do square ice and Izergin-Korepin-type determinants generalize to other solutions of the Yang-Baxter equation*?

Although our simultaneous treatment of several classes of ASMs is not especially short, the argument for any one alone is relatively simple. We speculate that the methods of Lagrange interpolation (used in §4) and factor exhaustion (the topic of §5) simplify many proofs of product formulas. I. J. Good's short proof of Dyson's conjecture [11] also uses Lagrange interpolation.

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