Shell games with 2×2 matrices

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Includes joint work with Josh Lam and Aaron Landesman.

Introduction

Some questions about *n*-tuples of matrices

Existence

$$A_1 A_2 \cdots A_n = \mathrm{id}$$

$$A_i \in GL_r(\mathbb{C})$$

Deligne-Simpson problem

Given conjugacy classes $C_1, \cdots, C_n \subset GL_r(\mathbb{C})$, do there exist matrices $A_1, \cdots, A_n$ with $A_i \in C_i$ such that

$$\prod A_i = \mathrm{id}?$$

Originally posed by Deligne; (partial) solution by Simpson, Crawley-Boevey, Kostov, $\cdots$

Uniqueness

Rigid local systems

Classify tuples of conjugacy classes $C_1, \cdots, C_n \subset GL_r(\mathbb{C})$, such that $(A_1, \cdots, A_n)$ with $A_i \in C_i$ and

$$\prod A_i = \text{id}$$

is *unique up to simultaneous conjugation*. Such tuples are called *rigid*.

Solved by Katz (1995).
Middle convolution:

$$C_1, \cdots, C_n \subset GL_r \rightsquigarrow C'_1, \cdots, C'_n \subset GL_{r'}$$

- If $(C_1, \cdots, C_n)$ is rigid, so is $(C'_1, \cdots, C'_n)$.
- If $(C_1, \cdots, C_n)$ is rigid and $r > 1$, then, $r' < r$.
- Middle convolution is invertible.

Canonical solutions

Canonical solutions

What are the canonical (conjugacy classes of) solutions to the equation

$$\prod A_i = \text{id?}$$

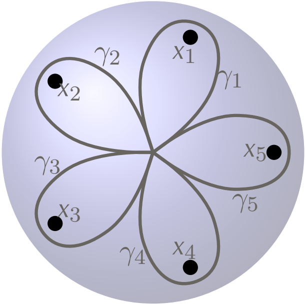
Canonical solutions

What are the finite orbits of $\langle \sigma_1, \cdots, \sigma_{n-1} \rangle$ on the set of (simultaneous) conjugacy classes of solutions to

$$A_1 A_2 \cdots A_n = \text{id?}$$

$$\sigma_i : (A_1, \cdots, A_n) \mapsto (A_1, \cdots, A_{i+1}, A_{i+1}^{-1} A_i A_{i+1}, \cdots, A_n).$$

$\mathbb{CP}^1 \setminus \{x_1, \cdots, x_n\}$



$$\pi_1(\mathbb{CP}^1 \setminus \{x_1, \cdots, x_n\}) = \langle \gamma_1, \cdots, \gamma_n \mid \prod_i \gamma_i = 1 \rangle$$

$$\rho : \pi_1(\mathbb{CP}^1 \setminus \{x_1, \cdots, x_n\}) \to \mathrm{GL}_r(\mathbb{C})$$

is the same as

$$A_1, \cdots, A_n \in \mathrm{GL}_r(\mathbb{C})^n \text{ such that } \prod_{i=1}^{n} A_i = \mathrm{id}.$$

The spherical braid group

$$\langle \sigma_1, \cdots, \sigma_{n-1} \rangle \approx \pi_0(\mathrm{Homeo}^+(S^2 \setminus \{x_1, \cdots, x_n\}))$$
$$\approx \pi_1(\mathrm{Conf}^n(S^2))$$
$$\approx \pi_1(\mathscr{M}_{0,n})$$

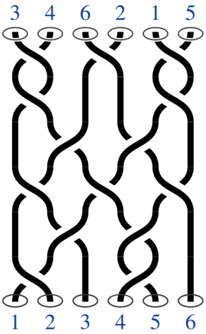
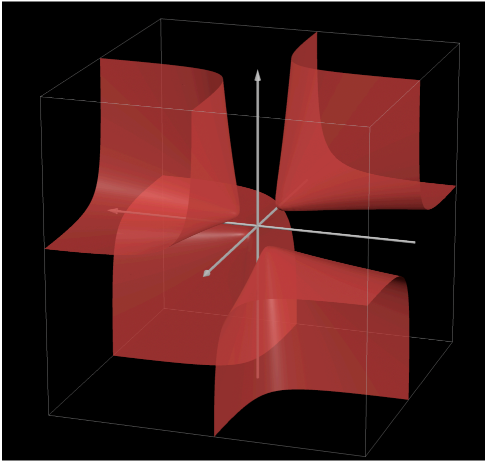


Image courtesy of nLab

Some history

The Markoff cubic



$$x^2 + y^2 + z^2 - 3xyz = 0$$

The Painlevé VI equation (Fuchs, 1905)

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt}$$
$$+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)$$

Here y is a function of t and $\alpha, \beta, \gamma, \delta$ are constants.

The Schlesinger system

$$\begin{cases} \dfrac{\partial B_i}{\partial \lambda_j} = \dfrac{[B_i, B_j]}{\lambda_i - \lambda_j} & \text{if } i \neq j \\ \displaystyle\sum_{i=1}^{n} \dfrac{\partial B_i}{\partial \lambda_j} = 0 & \forall j \end{cases}$$

where the $B_i, i = 1, \cdots, n$ are $\mathrm{Mat}_{r \times r}(\mathbb{C})$-valued functions in $(\lambda_1, \cdots, \lambda_n)$.
Painlevé VI is the case $n = 4$, where the B_i are $\mathfrak{sl}_2$-valued.

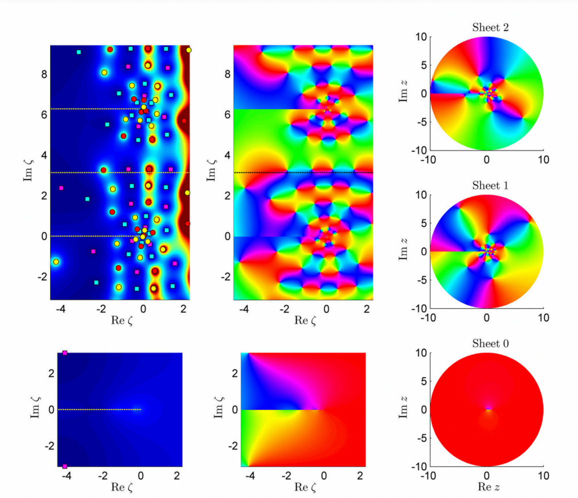
Key connection

Algebraic solutions correspond to finite orbits of $\langle \sigma_1, \cdots, \sigma_{n-1} \rangle$ on the set of conjugacy classes of solutions to

$$A_1 A_2 \cdots A_n = \mathrm{id},$$

with $A_i \in GL_r(\mathbb{C})$.

Painlevé Transcendents



From: Methods for the computation of the multivalued Painlevé transcendents on their Riemann surfaces, by Fasondini, Fornberg, Weideman, *Journal of Computational Physics*, 2017

Classification I (Dubrovin-Mazzocco)

The Puiseux expansions near $x = 1$ and $x = \infty$ can be obtained from these formulae applying the symmetries (1.22) and (1.23) respectively. Using these formulae, one can compute any term of the Puiseux expansions of all the branches. Due to computer difficulties, at the moment, we do not manage to produce the explicit elliptic parameterization of the algebraic curve. We give this in the form of an algebraic curve of degree 36.

$$x^{15} F(x,y,t) = 0 \qquad (H_3)''$$

where

$$t = x + \frac{1}{x}$$

and

$$\begin{aligned} F(x,y,t) = \big(&11423613917539180989 - 571698137302039944\,t - 13869163074392577\,t^2 \\ &+1307302991918736\,t^3 - 31962210377\,t^4 - 556854952\,t^5 + 282475249\,t^6\big)^2 x^9 + \\ &+9\,(-42194267411458338799378785573556538817 \\ &-58759262104428568315429822622247510492\,t+ \\ &+10095266581644469686796601774497789110\,t^2 \\ &-9698051065970388294721532496471607800\,t^3 \\ &+1308223958339537358154544139962717 7\,t^4 \\ &-770584465498507451654409567734 16\,t^5 \\ &-21505995316324737352252761967 88\,t^6 \\ &+5521397776112060589691860200\,t^7 \\ &+3443168943013224269825 6649\,t^8 - 48683795393280052041267 48\,t^9 \\ &+5432989909979975465 90\,t^{10} - 5420393254540081020\,t^{11} \end{aligned}$$

...and this goes on for 9 more pages

From: Dubrovin, B., Mazzocco, M. Monodromy of certain Painlevé–VI transcendents and reflection groups. Invent. math. 141, 55–147 (2000). https://doi.org/10.1007/PL00005790

Classification II (Lisovyy-Tykhyy)

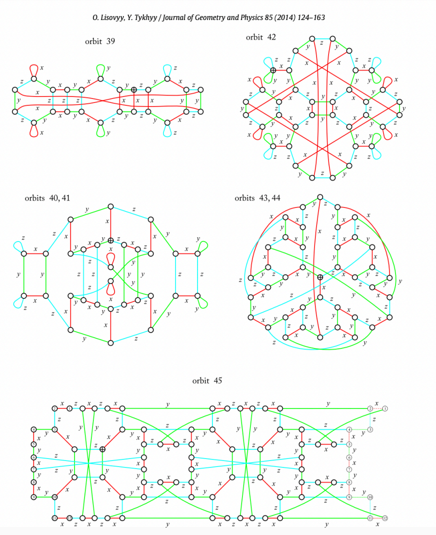


Fig. 11. Graphs of exceptional orbits 39–45.

From: Oleg Lisovyy, Yuriy Tykhyy, Algebraic solutions of the sixth Painlevé equation, Journal of Geometry and Physics, Volume 85, 2014, Pages 124-163, ISSN 0393-0440, https://doi.org/10.1016/j.geomphys.2014.05.010.

Examples

Four 2×2 matrices:

$$A_1 = \begin{pmatrix} 1 + x_2 x_3 / x_1 & -x_2^2 / x_1 \\ x_3^2 / x_1 & 1 - x_2 x_3 / x_1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & -x_1 \\ 0 & 1 \end{pmatrix},\ A_3 = \begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix},$$

$$A_4 = (A_1 A_2 A_3)^{-1}$$

where

$$x_1 = 2\cos\left(\frac{\pi(\alpha+\beta)}{2}\right),\ x_2 = 2\sin\left(\frac{\pi\alpha}{2}\right),\ x_3 = 2\sin\left(\frac{\pi\beta}{2}\right)$$

for $\alpha, \beta \in \mathbb{Q}$.
Five 2×2 matrices:

$$A_1 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, A_2 = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}, A_3 = \begin{pmatrix} \frac{-1-\sqrt{5}}{2} & 1 \\ \frac{-3+\sqrt{5}}{2} & \frac{-3+\sqrt{5}}{2} \end{pmatrix},$$

$$A_4 = \begin{pmatrix} \frac{1-\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} \\ \frac{-3+\sqrt{5}}{2} & \frac{-5+\sqrt{5}}{2} \end{pmatrix}, A_5 = (A_1 A_2 A_3 A_4)^{-1}$$

Classifying finite orbits

The goal

Classify finite orbits of the $\mathrm{Mod}_{0,n} = \langle \sigma_1, \cdots, \sigma_{n-1} \rangle$ action on

$$\{(A_1, \cdots, A_n) \in \mathrm{SL}_2(\mathbb{C})^n \text{ s.t. } \prod_{i=1}^{n} A_i = \mathrm{id}\}/\text{simultaneous conjugation.}$$

$$\sigma_i : (A_1, \cdots, A_n) \mapsto (A_1, \cdots, A_{i+1}, A_{i+1}^{-1} A_i A_{i+1}, \cdots, A_n).$$

The classification

Theorem (Lam-Landesman-L–, Bronstein-Maret, 2023-2024)

Suppose $(A_1, \cdots, A_n)$ is an interesting finite orbit. With one exception there exists explicit $\alpha_1, \cdots, \alpha_n, \lambda \in \mathbb{C}^\times$ such that

$$(\alpha_1 A_1, \cdots, \alpha_n A_n) = MC_\lambda(B_1, \cdots, B_n),$$

with $\langle B_1, \cdots, B_n \rangle \subset GL_{n-2}(\mathbb{C})$ a finite complex reflection group.

Interesting

$(A_1, \cdots, A_n)$ is an *interesting* finite orbit if:

- No $A_i = \pm\mathrm{id}$.
- $\langle A_1, \cdots A_n \rangle$ is Zariski-dense in $SL_2(\mathbb{C})$, i.e. it is not
  - finite (these were classified by Euclid), or
  - conjugate to

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \text{ or } \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \cup \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$$

(these were classified by Cousin-Moussard, McMullen, Tykhyy)

- $\langle A_1, \cdots A_n \rangle$ doesn't move in a continuous family of finite orbits (such continuous families classified by Corlette-Simpson, Doran, Diarra)

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Suppose $(A_1, \cdots, A_n)$ *is an interesting finite orbit. With one exception there exists explicit* $\alpha_1, \cdots, \alpha_n, \lambda \in \mathbb{C}^\times$ *such that*

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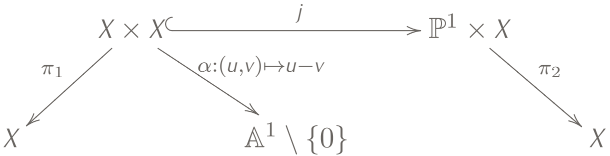
Middle convolution

Middle convolution is an algebro-geometric construction.

$$X = \mathbb{P}^1 \setminus \{x_1, \cdots, x_{n-1}, \infty\}$$



$$MC_\lambda(\mathbb{V}) = R^1\pi_{2*}j_*(\pi_1^*\mathbb{V} \otimes \alpha^*\chi_\lambda)$$

.

Explicit formula by Volklein, Dettweiler-Reiter. Implemented on a computer by Amal Vayalinkal.

Qualitative description

If λ is an m-th root of unity and $G = \langle B_1, \cdots, B_n\rangle$ is finite, constructs a family of Riemann surfaces $\pi : X \to \mathbb{CP}^1 \setminus \{x_1, \cdots, x_n\}$, with $\pi^{-1}(z)$ a $G \times \mathbb{Z}/m\mathbb{Z}$-cover of $\mathbb{CP}^1$ branched over $\{x_1, \cdots, x_n, z\}$.

The classification

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Suppose $(A_1, \cdots, A_n)$ *is an interesting finite orbit. With one exception there exists explicit* $\alpha_1, \cdots, \alpha_n, \lambda \in \mathbb{C}^\times$ *such that*

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with $\langle B_1, \cdots, B_n \rangle \subset GL_{n-2}(\mathbb{C})$ *a finite complex reflection group.*

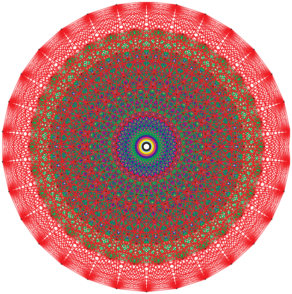
Finite complex reflection groups

Pseudoreflections

$B \in GL_r(\mathbb{C})$ is a *pseudoreflection* if it has finite order and $\mathrm{rk}(B - \mathrm{id}) = 1$.

Finite complex reflection group

A *finite complex reflection group* is a finite subgroup of $GL_r(\mathbb{C})$ generated by pseudoreflections.



A projection of the E_8 root lattice, generated by John Stembridge

The classification

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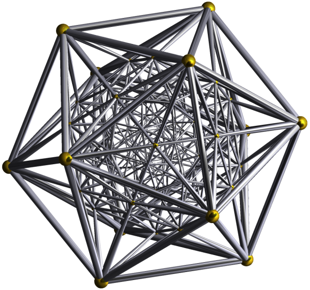
The fine classification

Shephard and Todd classified finite complex reflection groups in 1954.



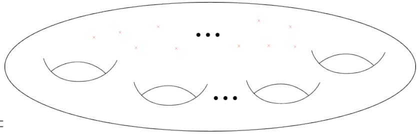
The 600-cell, from Wikimedia Commons

Corollary

Suppose $(A_1, \cdots, A_n)$ *is interesting. Then* $n \leq 6$.

Higher genus

The setup

$\Sigma_{g,n} :=$ 

$$\pi_1(\Sigma_{g,n}) = \langle a_1, b_1, \cdots, a_g, b_g, c_1, \cdots, c_n \mid \prod_{i=1}^{g}[a_i, b_i]\prod_{j=1}^{n} c_j\rangle$$

$$\begin{aligned}\mathrm{Mod}_{g,n} \quad &= \quad \pi_0(\mathrm{Homeo}^+(\Sigma_{g,n}))\\ &\quad\ \text{contains with finite index } \pi_1(\mathscr{M}_{g,n})\\ &\quad\ \text{if } n=0\text{, finite index in } \mathrm{Out}(\pi_1(\Sigma_g))\end{aligned}$$

The basic question

What are the finite orbits of the $\mathrm{Mod}_{g,n}$-action on $\mathrm{Hom}(\pi_1(\Sigma_{g,n}), \mathrm{GL}_r(\mathbb{C}))/\sim$?

Two conjectures

Consequence of conjecture of Esnault-Kerz, Budur-Wang

Finite orbits of the Mod_g-action on $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{GL}_r(\mathbb{C}))/\sim$ are Zariski-dense in $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{GL}_r(\mathbb{C}))/\sim$.

Question/Conjecture of Kisin and Whang

For $g \gg r$, finite orbits of the $\mathrm{Mod}_{g,n}$-action on $\mathrm{Hom}(\pi_1(\Sigma_{g,n}), \mathrm{GL}_r(\mathbb{C}))/\sim$ correspond exactly to representations with finite image.

These conjectures contradict each other!

Classification

Theorem (Landesman-L–, 2022)

Fix $g \geq r^2$, and let

$$\rho : \pi_1(\Sigma_{g,n}) \to GL_r(\mathbb{C})$$

be a representation whose conjugacy class has finite orbit under $Mod_{g,n}$. Then ρ has finite image.

Proof relies on non-Abelian Hodge theory and input from the Langlands program over function fields.
Due to Biswas, Gupta, Mj, and Whang when $r = 2$.

The general picture(?)