On the converse to Eisenstein's last theorem

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Includes joint work with Josh Lam.

When does a differential equation have algebraic solutions? (Fuchs, 1875)

Lazarus Fuchs,
Universitätsbibliothek Heidelberg

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$e^z, \log(z)$ are not algebraic

Hypergeometric functions

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \text{ where } (a)_n = a(a+1)(a+2)\cdots(a+n-1)$$

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No.	λ''	μ''	ν''	$\frac{\text{Inhalt}}{\pi}$	Polyeder
I.	$\frac{1}{2}$	$\frac{1}{2}$	ν	ν	Regelmässige Doppelpyramide
II.	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}=A$	Tetraeder
III.	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}=2A$	
IV.	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{12}=B$	Würfel und Oktaeder
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X.	$\frac{3}{5}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{2}{15}=4C$	
XI.	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{1}{5}=6C$	
XII.	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{5}=6C$	
XIII.	$\frac{4}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}=6C$	
XIV.	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{7}{30}=7C$	
XV.	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{1}{3}=10C$	

Schwarz's list (1973)

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Hermann Schwarz, Wikimedia commons

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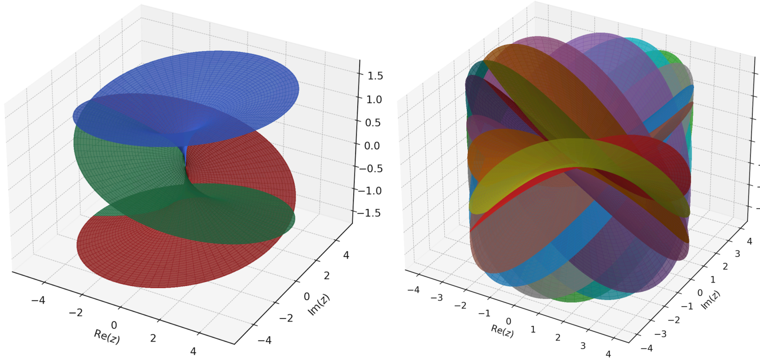
An example, cont.



Riemann surfaces for $z^{1/3}$ and $z^{\sqrt{2}}$

The Grothendieck-Katz p-curvature conjecture

Conjecture (Grothendieck-Katz)

Let $A \in \mathrm{Mat}_{r\times r}(\overline{\mathbb{Q}}(z))$. All solutions to

$$\left(\frac{d}{dz} - A\right)\vec{f}(z) = 0$$

are algebraic $\iff$

$$\left(\frac{d}{dz} - A\right)^p \equiv 0 \bmod p$$

for all but finitely many primes p.

An example, continued

Want to show:

$$\left(\frac{d}{dz} - \frac{a}{z}\right)^p \equiv 0 \bmod p$$

for almost all primes p iff a is rational.

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Known cases

The Grothendieck-Katz p-curvature conjecture is known for:

- Picard-Fuchs equations (Katz, 1972)
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Beautiful work by Esnault-Kisin, Esnault-Groechenig, Farb-Kisin, Shankar, Shankar-Patel-Whang, Tang, $\cdots$

What about...?

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- What about a more naive criterion?
  - (Stanley) Is there a criterion in terms of the coefficients of the Taylor expansion of a solution?

Eisenstein's last theorem

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Gotthold Eisenstein

A mystery

von beliebig vielen algebraischen Gleichungen hervorgehenden Entwicklungen auszudehnen. Die wichtigsten Anwendungen der so erhaltenen Sätze habe ich auf Fälle gemacht, in denen die algebraischen Funktionen als Integrale von Differential-Gleichungen definirt werden, und diese Differential-Gleichungen für einfache Reihen-Entwicklung geeignet sind, während die vielleicht sehr complicirte Darstellung in endlicher Form ganz unbekannt bleibt und für diesen Zweck auch wirklich ganz aus dem Spiele gelassen werden kann. Das Einzelne der hieranf bezüglichen Untersuchungen mag für eine künftige Mittheilung vorbehalten bleiben. Eine andere sehr einfache Art der Anwendung beruht darauf, … mit rationalen …

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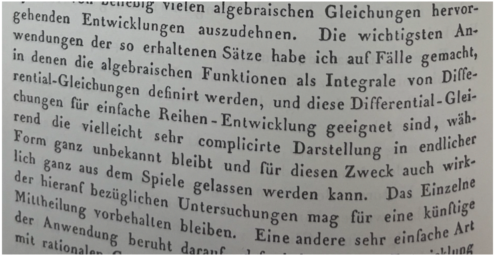

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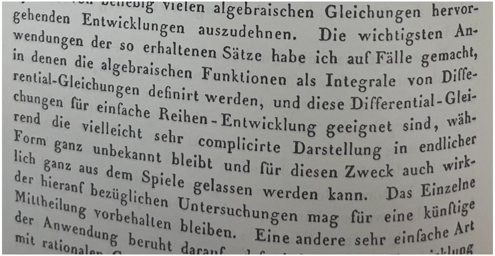

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$$F(z) = \frac{1}{\pi}\int_0^1 \frac{dx}{\sqrt{x(1-x)(1-zx)}} = \sum_{n=0}^{\infty} \binom{2n}{n}^2 \left(\frac{z}{16}\right)^n$$

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3. There exists ω : Primes $\to \mathbb{Z}$ with $\lim_{p\to\infty} \frac{\omega(p)}{p} = \infty$ such that

   $$a_0, a_1, \cdots, a_{\omega(p)} \in \mathbb{Z}_{(p)}$$

   for almost all p.

Results

Elliptic integrals, modular forms, ...

Theorem (Lam-L–)

Let $z_0 \in \mathbb{Q} \setminus \{0,1\}$. Suppose $F(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is non-zero and satisfies

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Example

$$F(z) = 1 + \frac{9}{16}\left(z - \frac{1}{3}\right)^2 + \cdots - \frac{3^{27} \cdot 2071973 \cdot 584141735992051147}{2^{67} \cdot 5 \cdot 7 \cdot 13 \cdot 23 \cdot 29}\left(z - \frac{1}{3}\right)^{29} + \cdots$$

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$$\frac{\partial^3 G}{\partial t^3} + \frac{6\left(t^2-32\right)}{t\left(t^2-64\right)} \frac{\partial^2 G}{\partial t^2} + \frac{7t^2-64}{t^2\left(t^2-64\right)} \frac{\partial G}{\partial t} + \frac{1}{t\left(t^2-64\right)} G = 0.$$

Digression: differential equations in algebraic geometry

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Topology of fibers is locally constant (Ehresmann)

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Initial conditions: $H^k(X_0, \mathbb{C})$.

Example

$$\mathcal{D}F = 0$$

where

$$\begin{aligned}\mathcal{D} = {} & 121\left(z\tfrac{d}{dz}-1\right)\left(z\tfrac{d}{dz}\right)^5-22z\left(438\left(z\tfrac{d}{dz}\right)^5+2094\left(z\tfrac{d}{dz}\right)^4+1710\left(z\tfrac{d}{dz}\right)^3+950\left(z\tfrac{d}{dz}\right)^2+275\left(z\tfrac{d}{dz}\right)+33\right)\left(z\tfrac{d}{dz}\right)\\
& -z^2\left(839313\left(z\tfrac{d}{dz}\right)^6+2471661\left(z\tfrac{d}{dz}\right)^5+4037556\left(z\tfrac{d}{dz}\right)^4+4497304\left(z\tfrac{d}{dz}\right)^3+3093948\left(z\tfrac{d}{dz}\right)^2+1158740\left(z\tfrac{d}{dz}\right)+180048\right)\\
& -2z^3\left(5746754\left(z\tfrac{d}{dz}\right)^6+26470666\left(z\tfrac{d}{dz}\right)^5+51184224\left(z\tfrac{d}{dz}\right)^4+50480470\left(z\tfrac{d}{dz}\right)^3+26295335\left(z\tfrac{d}{dz}\right)^2+6684843\left(z\tfrac{d}{dz}\right)+604098\right)\\
& -4z^4\left(4081884\left(z\tfrac{d}{dz}\right)^6+14894484\left(z\tfrac{d}{dz}\right)^5+18825903\left(z\tfrac{d}{dz}\right)^4+7472030\left(z\tfrac{d}{dz}\right)^3-3698839\left(z\tfrac{d}{dz}\right)^2-4099839\left(z\tfrac{d}{dz}\right)-993618\right)\\
& +56z^5\left(29592\left(z\tfrac{d}{dz}\right)^6+255960\left(z\tfrac{d}{dz}\right)^5+806448\left(z\tfrac{d}{dz}\right)^4+1272787\left(z\tfrac{d}{dz}\right)^3+1088403\left(z\tfrac{d}{dz}\right)^2+483431\left(z\tfrac{d}{dz}\right)+87609\right)\\
& \quad + 1568z^6\left(4z\tfrac{d}{dz}+5\right)\left(2z\tfrac{d}{dz}+3\right)\left(4z\tfrac{d}{dz}+3\right)\left(z\tfrac{d}{dz}+1\right)^3.\end{aligned}$$

What is a linear differential equation?

Let X be a complex manifold. Analytic continuation of solutions gives an equivalence of categories

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Upshot: If you have a way of associating a vector space to a complex manifold, it should give you linear differential equations.

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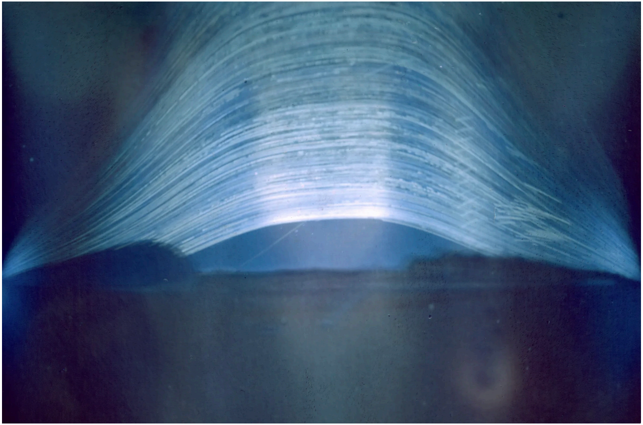

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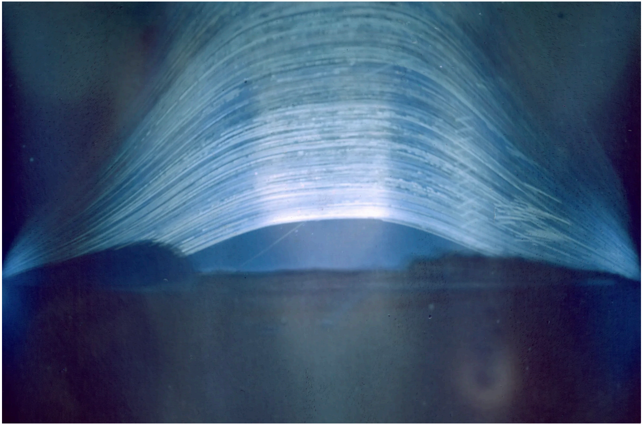

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Answer

A differential equation is a foliation.

Where do differential equations come from?

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- Solutions: Families of flat bundles $(\mathscr{E}_s, \nabla_s)$ (differential equations) whose associated monodromy representation isn't changing.

Example of isomonodromy ODE

$X_s = \mathbb{CP}^1 \setminus \{\lambda_1, \cdots, \lambda_n\}$; ODE with regular singularities:

$$\left(\frac{d}{dz} + \sum \frac{A_i}{z - \lambda_i} \right) \vec{f}(z) = 0 \text{ where } A_i \in \mathrm{Mat}_{r \times r}(\mathbb{C}), \sum_i A_i = 0$$

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Answer: the Schlesinger system

$$\begin{cases} \dfrac{\partial A_i}{\partial \lambda_j} = \dfrac{[A_i, A_j]}{\lambda_i - \lambda_j} & \text{if } i \neq j \\ \displaystyle\sum_{i=1}^{n} \dfrac{\partial A_i}{\partial \lambda_j} = 0 \end{cases}$$

Results

Picard-Fuchs equations

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Theorem (Lam-L–)

Suppose $F(s) = \int_\sigma \omega_s$ is the (formal) solution to a Picard-Fuchs equation corresponding to $\sigma_0 \in H^{2k}(X_0, \mathbb{C})$, with σ_0 the class of an algebraic cycle of codimension k. Then the following are equivalent:

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Example

$$\mathcal{D}F = 0$$

where

$$\begin{aligned}\mathcal{D} = {} & 121\left(z\tfrac{d}{dz}-1\right)\left(z\tfrac{d}{dz}\right)^5-22z\left(438\left(z\tfrac{d}{dz}\right)^5+2094\left(z\tfrac{d}{dz}\right)^4+1710\left(z\tfrac{d}{dz}\right)^3+950\left(z\tfrac{d}{dz}\right)^2+275\left(z\tfrac{d}{dz}\right)+33\right)\left(z\tfrac{d}{dz}\right)\\
& -z^2\left(839313\left(z\tfrac{d}{dz}\right)^6+2471661\left(z\tfrac{d}{dz}\right)^5+4037556\left(z\tfrac{d}{dz}\right)^4+4497304\left(z\tfrac{d}{dz}\right)^3+3093948\left(z\tfrac{d}{dz}\right)^2+1158740\left(z\tfrac{d}{dz}\right)+180048\right)\\
& -2z^3\left(5746754\left(z\tfrac{d}{dz}\right)^6+26470666\left(z\tfrac{d}{dz}\right)^5+51184224\left(z\tfrac{d}{dz}\right)^4+50480470\left(z\tfrac{d}{dz}\right)^3+26295335\left(z\tfrac{d}{dz}\right)^2+6684843\left(z\tfrac{d}{dz}\right)+604098\right)\\
& -4z^4\left(4081884\left(z\tfrac{d}{dz}\right)^6+14894484\left(z\tfrac{d}{dz}\right)^5+18825903\left(z\tfrac{d}{dz}\right)^4+7472030\left(z\tfrac{d}{dz}\right)^3-3698839\left(z\tfrac{d}{dz}\right)^2-4099839\left(z\tfrac{d}{dz}\right)-993618\right)\\
& +56z^5\left(29592\left(z\tfrac{d}{dz}\right)^6+255960\left(z\tfrac{d}{dz}\right)^5+806448\left(z\tfrac{d}{dz}\right)^4+1272787\left(z\tfrac{d}{dz}\right)^3+1088403\left(z\tfrac{d}{dz}\right)^2+483431\left(z\tfrac{d}{dz}\right)+87609\right)\\
& \quad +1568z^6\left(4z\tfrac{d}{dz}+5\right)\left(2z\tfrac{d}{dz}+3\right)\left(4z\tfrac{d}{dz}+3\right)\left(z\tfrac{d}{dz}+1\right)^3.\end{aligned}$$

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An example: Painlevé VI

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt}$$
$$+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)$$

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$$y(t) = 2 - \frac{8}{3}\left(z - \frac{1}{2}\right)^2 + \cdots + \frac{3323732992}{3^8 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13}\left(z - \frac{1}{2}\right)^{13} + \cdots$$

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Proof input

Picard-Fuchs equation:

- Hodge theory
- Fontaine-Lafaille theory

Isomonodromy differential equations:

- Non-abelian Hodge theory over $\mathbb{C}$ (Simpson, ...)
- Non-abelian Hodge theory in positive characteristic (Ogus-Vologodsky, Schepler)
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Key idea: arithmetic version of the variational Hodge conjecture (and its non-abelian analogue)

Relationship to the p-curvature conjecture

Proposition (Lam-L–, Lawrence-L–)

Proving the main conjecture for isomonodromy differential equations (and arbitrary initial conditions) implies the Grothendieck-Katz p-curvature conjecture in general.

The conjecture

Conjecture

Suppose $f(z) = \sum a_n z^n \in \mathbb{Q}[[z]]$ satisfies

$$f^{(n)} = G(z, f(z), \cdots, f^{(n-1)}(z)),$$

where $G \in \mathbb{Q}(z, y_0, \cdots, y_{n-1})$ is defined at $(0, f(0), \cdots, f^{(n-1)}(0))$. Then the following are equivalent:

1. f is algebraic over $\mathbb{Q}(z)$.
2. There exists N such that all $a_n \in \mathbb{Z}\left[\frac{1}{N}\right]$.
3. There exists ω : Primes $\to \mathbb{Z}$ with $\lim_{p\to\infty} \frac{\omega(p)}{p} = \infty$ such that

   $$a_0, a_1, \cdots, a_{\omega(p)} \in \mathbb{Z}_{(p)}$$

   for almost all p.