Pretzel knots admitting L-space surgeries

Tye Lidman ‡  Allison Moore†

†Rice University
‡University of Texas

October 13, 2013
Pretzel knots

$K \subset S^3$.

Pretzel knot $(n_1, \ldots, n_k)$,

$n_i \in \mathbb{Z}$.

$K = (5, -7, 5, -4)$
A familiar construction: Dehn surgery

\[ X = S^3 - \nu(K) \]

\[ H_1(X) = \langle [\mu] \rangle \]

\[ H_1(\partial X) = \langle [\mu], [\lambda] \rangle^{ab} \]

\[ V \to X \]

\[ \mu V \mapsto \alpha \]

\( S^3_{p/q}(K) \) is called \((p, q)\)-Dehn surgery along \( K \).

\[ \alpha = p\mu + q\lambda \]
Exercise:

\[ H_1(S^3_{p/q}(K)) = \mathbb{Z}/p\mathbb{Z}. \]

Lens spaces arise from this construction when \( K \) is the unknot.

\[ \alpha = p\mu + q\lambda \]
A longstanding question

Question

*Which knots admit lens space surgeries?*
A bit of history

Which knots admit lens space surgeries?

1971  (Moser) some torus knots.
A bit of history

Which knots admit lens space surgeries?

1971  (Moser) some torus knots.
1977  (Bailey-Rolfsen) iterated torus knot.
A bit of history

Which knots admit lens space surgeries?

1971 (Moser) some torus knots.
1977 (Bailey-Rolfsen) iterated torus knot.
1980 (Fintushel-Stern) hyperbolic knot \((-2, 3, 7)\).
A bit of history

Which knots admit lens space surgeries?

1971 (Moser) some torus knots.
1977 (Bailey-Rolfsen) iterated torus knot.
1980 (Fintushel-Stern) hyperbolic knot \((-2, 3, 7)\).
1990 (Berge) many examples.

Cyclic Surgery Theorem (CGLS) + Berge’s construction = ‘The Berge Conjecture.’
Knot Floer homology

(Ozsváth-Szabó, Rasmussen):

\[ K \subset M \leadsto \cdots \subset F_{i-1}C \subset F_iC \subset \ldots \]
\[ \leadsto H_*(F_iC/F_{i-1}C) \]

For me:

- \( M = S^3 \).
- Coefficients in \( \mathbb{F} = \mathbb{Z}/2\mathbb{Z} \).
- Simplest version.
- \( (m, s) \in \mathbb{Z} \oplus \mathbb{Z} \).

\[
\widehat{HFK}(K) = \bigoplus_{m,s} \widehat{HFK}_m(K, s).
\]

\[
\Delta_K(t) = \sum_s \chi(\widehat{HFK}(K, s)) \cdot t^s
\]
Fact:
\[ \text{rank } \hat{HF}(L(p, q)) = p = |H_1(L(p, q))|. \]

A \( \mathbb{Q}HS^3 \) \( Y \) is an **L-space** if
\[ |H_1(Y; \mathbb{Z})| = \text{rank } \hat{HF}(Y). \]

Examples: \( S^3 \), all lens spaces, 3-manifolds with finite \( \pi_1 \).
Motivating question recast

Question

*Which knots admit lens space surgeries?*

becomes

Question

*Which knots admit L-space surgeries?*
Theorem (Ozsváth-Szabó)

If $K$ admits an L-space surgery, then for all $s \in \mathbb{Z}$,

$$\widehat{HFK}(K, s) \cong \mathbb{F} \text{ or } 0.$$  

(and some other conditions on Maslov grading).
L-space surgeries

Theorem (Ozsváth-Szabó)

If $K$ admits an L-space surgery, then for all $s \in \mathbb{Z}$,

$$\widehat{HFK}(K, s) \cong \mathbb{F} \text{ or } 0.$$  

(and some other conditions on Maslov grading).

Corollary

If $K$ admits an L-space surgery, $|a_i| \leq 1$ for all coefficients $a_i$ of $\Delta_K(t)$.  

Allison Moore  Pretzel knots with L-space surgeries
Theorem (Lidman-M)

Let $K$ be a pretzel knot with any number of tangles. Then $K$ admits an L-space surgery if and only if

1. $K \simeq T(2, 2n + 1), \ n \in \mathbb{Z},$ or
2. $K \simeq \pm(-2, 3, q), \ q \geq 1 \in \mathbb{Z}$ odd.
Remark

1. \((-2, 3, 1) \cong T(2, 5)\)
2. \((-2, 3, 3) \cong T(3, 4)\)
3. \((-2, 3, 5) \cong T(3, 5)\)
Remark

1. $(-2, 3, 1) \simeq T(2, 5)$
2. $(-2, 3, 3) \simeq T(3, 4)$
3. $(-2, 3, 5) \simeq T(3, 5)$
4. $(-2, 3, 7)$: hyperbolic knot with two cyclic surgeries ($+18, +19$). (Fintushel-Stern).
5. $(-2, 3, 9)$: has two finite non-cyclic surgeries (Bleiler-Hodgson).
Remark

1. $(-2, 3, 1) \simeq T(2, 5)$
2. $(-2, 3, 3) \simeq T(3, 4)$
3. $(-2, 3, 5) \simeq T(3, 5)$
4. $(-2, 3, 7)$: hyperbolic knot with two cyclic surgeries ($+18$, $+19$). (Fintushel-Stern).
5. $(-2, 3, 9)$: has two finite non-cyclic surgeries (Bleiler-Hodgson).
6. $(-2, 3, q)$, $q > 11$ odd: $S^3_{2q+4}(K)$ is a Seifert fibered L-space (Ozsváth-Szabó).
Remark

1. \((-2, 3, 1) \cong T(2, 5)\)
2. \((-2, 3, 3) \cong T(3, 4)\)
3. \((-2, 3, 5) \cong T(3, 5)\)
4. \((-2, 3, 7)\): hyperbolic knot with two cyclic surgeries \((+18, +19)\). (Fintushel-Stern).
5. \((-2, 3, 9)\): has two finite non-cyclic surgeries (Bleiler-Hodgson).
6. \((-2, 3, q), \ q > 11 \text{ odd}: S^3_{2q+4}(K) \text{ is a Seifert fibered L-space} \) (Ozsváth-Szabó).
Proof sketch

It remains to show no other pretzel knot qualifies.
It remains to show no other pretzel knot qualifies.

**Theorem (Ni, Ghiggini)**

\[ K \text{ is fibered if and only if } \widehat{\text{HFK}}(K, g(K)) \cong \mathbb{F}. \]

**Theorem (Ozsváth-Szabó, Crowell-Murasugi)**

An alternating knot admits an L-space surgery if and only if 
\[ K \cong T(2, 2n + 1), \text{ some } n \in \mathbb{Z}. \]

Thus we only consider non-alternating, fibered pretzel knots.
Fibered pretzel links

Classified by Gabai in the 1980s.

Three types:
1. no $m_i$
2. both $m_i$ and $m_{ij}$
3. no $m_{ij}$.
Fibered pretzel links

Classified by Gabai in the 1980s.

Three types:
1. no $m_i$
2. both $m_i$ and $m_{ij}$.
3. no $m_{ij}$.

Within each type, we use two basic techniques to obstruct $K$ from admitting an L-space surgery.
1. Show $\det(K)$ is sufficiently large.
2. Compute coefficients of $\Delta_K(t)$ using the Kauffman state sum decomposition.
Lemma

If \( \det(K) > 2g(K) + 1 \), then \( K \) is not an L-space knot.
Lemma

If \( \det(K) > 2g(K) + 1 \), then \( K \) is not an L-space knot.

Proof.

If \( K \) is an L-space knot, then \( |a_s| \leq 1 \forall s \). Then,

\[
\det(K) = |\Delta_K(-1)| \leq \sum_s |a_s| \leq 2g(K) + 1.
\]
(2) A state sum for $\Delta_K(t)$

**Theorem (Kauffman)**  

*The Alexander polynomial admits a state sum formula*

$$\Delta_K(T) = \sum_{x \in S} (-1)^{M(x)} T^{A(x)},$$

*where $S$ is the set of Kauffman states of a decorated knot diagram, and $A$ and $M$ are maps*

$$A : S \to \mathbb{Z}, \quad M : S \to \mathbb{Z}.$$
Kauffman states

Let $G_B$ and $G_W$ be the black and white graphs associated with a checkerboard coloring of a decorated knot projection.

$$S \leftrightarrow \{ \text{spanning trees of } G_B \}$$
Basic idea

Apply mirroring, isotopy, and mutation to $K$ until $K^\tau$ admits a diagram with a unique state $x_0$ s.t.

$$A(x_0) = -g(K^\tau).$$

Use $x_0$ to count states $x$ such that

$$A(x) = -g(K^\tau) + 1,$$

and calculate $M(x)$ relative to $M(x_0)$. Then show

$$|a_{-g(K^\tau)+1}| = \sum_{A(x)=-g(K^\tau)+1} (-1)^{M(x)} > 1.$$

Because the Alexander polynomial is preserved by mutation, obtain $|a_s| > 1$ for some coefficient $a_s$ of $\Delta_K(t)$.
Corollary (Ichihara-Jong, Lidman-M.)

The only nontrivial pretzel knots admitting nontrivial finite surgeries are $\pm(-2, 3, q)$ for $q = 1, 3, 5, 7, 9$.

Observation

Up to mirroring, $\exists!$ fibered pretzel knot with the Alexander polynomial of an L-space knot, but no L-space surgery. This is

$$K = (3, -5, 3, -2).$$
Observe that no pretzel knot admitting an L-space surgery contains an essential Conway sphere in its complement.

**Question**

*Can an L-space knot have an essential Conway sphere?*

**Question**

*Do L-space knots have any nontrivial mutants?*
Observe that no pretzel knot admitting an L-space surgery contains an essential Conway sphere in its complement.

**Question**

*Can an L-space knot have an essential Conway sphere?*

**Question**

*Do L-space knots have any nontrivial mutants?*

**Question**

*Can an lens space knot have an essential Conway sphere?*

**Answer:** No (Wu).
Observe that no pretzel knot admitting an L-space surgery contains an essential Conway sphere in its complement.

Question

Can an L-space knot have an essential Conway sphere?

Question

Do L-space knots have any nontrivial mutants?

Question

Can an lens space knot have an essential Conway sphere?

Answer: No (Wu).

Question

Can the knot Floer complex “see” essential Conway spheres?
Thank you!
Conjecturally, the total rank of knot Floer homology is invariant under (genus two) mutation. Evidence:

1. All knots through 12 crossings.
2. Many computed examples of 13 and 14 crossings.
3. Related conjecture of Baldwin-Levine about $\delta$-graded $\widehat{\text{HFK}}$.
4. And:

**Theorem (M. and Starkston)**

*There exist infinitely many genus two mutant pairs $K_n, K_n^\tau$ with distinct knot Floer homology of the same total dimension.*
Assuming total rank is preserved under mutation:

If true, then given an L-space knot $K$, and any mutant $K^\tau$ of $K$,

$$\widehat{\text{HFK}}_m(K, s) \cong \widehat{\text{HFK}}_m(K^\tau, s).$$

This suggests that $K \simeq K^\tau$, i.e., that L-space knots admit only trivial mutations.
The red-herring pretzel \((3, -5, 3, -2)\)

<table>
<thead>
<tr>
<th>(M)</th>
<th>(\Delta_K(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>(F^4)</td>
</tr>
<tr>
<td>3</td>
<td>(F^3)</td>
</tr>
<tr>
<td>2</td>
<td>(F^4)</td>
</tr>
<tr>
<td>1</td>
<td>(F^3)</td>
</tr>
<tr>
<td>0</td>
<td>(F^4)</td>
</tr>
<tr>
<td>-1</td>
<td>(F^3)</td>
</tr>
<tr>
<td>-2</td>
<td>(F, F^2)</td>
</tr>
<tr>
<td>(A)</td>
<td>(-3, -2, -1, 0, 1, 2, 3)</td>
</tr>
</tbody>
</table>