Montesinos knots, Hopf plumbings and L-space surgeries

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A longstanding question

Which knots admit lens space surgeries?

1971 (Moser)
1977 (Bailey-Rolfsen)
1980 (Fintushel-Stern)
1990 (Berge)

\[ \alpha = p\mu + q\lambda \]

Cyclic Surgery Theorem (CGLS) + Berge’s construction
= “The Berge Conjecture.”
L-spaces

(Ozsváth-Szabó, Rasmussen): Knot Floer homology.

\[ K \subset Y \hookrightarrow \cdots \subset \mathcal{F}_{i-1} C \subset \mathcal{F}_i C \subset \cdots \]

\[ H_\ast(\mathcal{F}_i C/\mathcal{F}_{i-1} C) \]

\[ \hat{\text{HFK}}(K) = \bigoplus_{m,s} \hat{\text{HFK}}_m(S^3, K, s). \]

- \[ \Delta_K(t) = \sum_s \chi(\hat{\text{HFK}}(K, s)) \cdot t^s \]
- A \( \mathbb{Q} HS^3 \) \( Y \) is an **L-space** if \( |H_1(Y; \mathbb{Z})| = \text{rank} \hat{\text{HF}}(Y) \).
  Ex: \( S^3 \), all lens spaces, 3-manifolds with finite \( \pi_1 \).
Motivating question revisited

Question

*Which knots admit lens space surgeries?*

becomes

Question

*Which knots admit L-space surgeries?*
Theorem (Ozsváth-Szabó)

*If* $K$ *admits an L-space surgery, then for all* $s \in \mathbb{Z}$,

$\widehat{HF}(K, s) \cong \mathbb{F}$ *or 0 (and some other conditions on Maslov grading).*
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\[ \hat{\text{HFK}}(K, s) \cong \mathbb{F} \text{ or } 0 \] (and some other conditions on Maslov grading).

Corollary (Determinant-genus inequality)

If \( \det(K) > 2g(K) + 1 \), then \( K \) is not an L-space knot.
L-space surgery obstructions

Theorem (Ozsváth-Szabó)

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\[
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\]

Corollary (Determinant-genus inequality)

If \( \det(K) > 2g(K) + 1 \), then \( K \) is not an L-space knot.

Proof.

If \( K \) is an L-space knot, then \( |a_s| \leq 1 \ \forall \ \text{coefficients } a_s \text{ of } \Delta_K(t) \).

Then,
\[
\det(K) = |\Delta_K(-1)| \leq \sum_s |a_s| \leq 2g(K) + 1.
\]
Theorem (Ni, Ghiggini)

\( K \) is fibered if and only if \( \hat{\text{HFK}}(K, g(K)) \cong F \).

Thus \( L \)-space knots are fibered.
More geometric obstructions

**Theorem (Ni, Ghiggini)**

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Thus \( L \)-space knots are fibered.

**Theorem (Hedden)**

An \( L \)-space knot \( K \) supports the tight contact structure; equivalently, an \( L \)-space knot is strongly quasipositive.
Theorem (Baker-M.)

Among the Montesinos knots, the only L-space knots are

- the pretzel knots $P(-2, 3, 2n + 1)$ for $n \geq 0$,
- and the torus knots $T(2, 2n + 1)$ for $n \geq 0$. 

Montesinos knots

\[ K = M\left(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \ldots, \frac{\beta_r}{\alpha_r} \mid e\right) \]

Figure: \( M(\frac{3}{4}, -\frac{2}{5}, \frac{1}{3} \mid 3) \).

Where \( \alpha_i, \beta_i, e \in \mathbb{Z} \) and \( \alpha_i > 1, |\beta_i| < \alpha_i \), and \( \gcd(\alpha_i, \beta_i) = 1 \).
Ingredients for proof

We need only consider fibered, non-alternating Montesinos knots,

\[ K = M \left( \frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \ldots, \frac{\beta_r}{\alpha_r} \mid e \right) \]

and we assume \( r \geq 3 \), because \( r \leq 2 \) implies \( K \) is a two-bridge link.

**Theorem (Ozsváth-Szabó)**

*An alternating knot admits an L-space surgery if and only if \( K \simeq T(2, 2n + 1) \), some \( n \in \mathbb{Z} \).*
(Hirasawa-Murasugi): Classified fibered Montesinos knots with their fibers. For $K = M\left(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \ldots, \frac{\beta_r}{\alpha_r} \mid e\right)$,

$$\frac{\beta_i}{\alpha_i} = \frac{1}{x_1 - \frac{1}{x_2 - \frac{1}{\ddots - \frac{1}{x_m}}}}$$

$$S_i := \left[x_1, \ldots, x_m\right]$$

have two cases of $S_i$:

1. $\alpha_i$ are all odd $\rightarrow$ strict continued fractions.
2. $\alpha_1$ is even, $\alpha_i$ is odd for $i > 1$ $\rightarrow$ even continued fractions.
Example: odd case

Each $\beta_i/\alpha_i$ has a strict continued fraction:

$$S_i = [2a_1^{(i)}, b_1^{(i)}, \ldots, 2a_q^{(i)}, b_q^{(i)}]$$

Hirasawa-Mursagi give strong restrictions on $e, S_1, \ldots, S_m$ when $M$ is fibered.

Figure: Image of odd-type Seifert surface borrowed from Hirasawa-Murasugi.
Open books for three-manifolds

\((F, \phi)\) — an open book for closed 3-manifold \(Y\).

\(L = \partial F\) is the binding.

\(F\) is the fiber surface.
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$(F, \phi)$ — an open book for closed 3-manifold $Y$.

$L = \partial F$ is the binding.

$F$ is the fiber surface.

$\xi$ — a contact structure on $Y$.

- Locally, $\ker \alpha$, $\alpha \wedge d\alpha \neq 0$
- (Thurston-Winkelnkemper - 1975) Every $(F, \phi)$ induces a contact structure.
- (Giroux - 2000) \{or. $\xi$ on $Y$\}/ isotopy $\leftrightarrow$ \{(F, \phi) for $Y$\} / positive stabilization
Plumbings of Hopf bands

Hopf links:

- $L^+ = \{(z_1, z_2) \in S^3 \subset \mathbb{C}^2 | z_1z_2 = 0\}$.
- $L^- = \{(z_1, z_2) \in S^3 \subset \mathbb{C}^2 | z_1\overline{z_2} = 0\}$.

Pos/neg (de)stabilization $\leftrightarrow$ (de)plumbing of pos/neg Hopf bands.

Figure: The connected sum of a positive and negative Hopf band.
Lemma (Contact Structures Lemma)

1. *(Goodman)*: 
   If $F \supset H_-$, then $\xi(F, \phi)$ is overtwisted.
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Lemma (Contact Structures Lemma)

1. (Goodman):
   If $F \supset H_-$, then $\xi(F,\phi)$ is overtwisted.

2. (Yamamoto):
   If $F$ contains a twisting loop, then $\xi(F,\phi)$ is overtwisted.

3. (Giroux):
   If $F \supset H_+$ and
   $$(F, \phi) = (F', \phi') \ast (H_+, \pi^+)$$
   then
   $$\xi(F,\phi) \cong \xi(F',\phi').$$
**Theorem (Baker-M.)**

A fibered Montesinos knot that supports the tight contact structure is isotopic to either

\[ M\left(\frac{-d_1}{2d_1+1}, \frac{-d_2}{2d_2+1}, \ldots, \frac{-d_r}{2d_r+1}\middle|1\right) \]

\[ M\left(\frac{-m_1}{m_1+1}, \frac{-m_2}{m_2+1}, \ldots, \frac{-m_r}{m_r+1}\middle|2\right) \]

**Figure:** Left: odd type. Right: even type.

And its fiber is obtained from the disk by a sequence of Hopf plumbings.
Odd case

- Repeatedly apply the Contact Structures Lemma, parts 1 & 2 to identify negative Hopf bands and/or twisting loops.
- Cull these knots because they support an overtwisted contact structure.

\[ b_{j-1}^{(i)} > 0 \]
\[ 2a_j^{(i)} = 2 \]
\[ b_j^{(i)} < 0 \]

**Figure:** Finding negative Hopf bands in \( F \).
Odd case

- Odd fibered Montesinos knots without a $H^-$ remain.
- Successively deplumb $H^+$ until a single $H^+$ remains.
- These knots support the tight contact structure.

$M\left(\frac{-d_1}{2d_1+1}, \frac{-d_2}{2d_2+1}, \ldots, \frac{-d_r}{2d_r+1} | 1\right)$
Lemma

Let $K$ be an odd fibered Montesinos knot supporting the tight contact structure. Then $\det(K) > 2g(K) + 1$ unless $K = M(\frac{1}{3}, \frac{1}{3}, \frac{2}{5}|1)$.

For any $K = M\left(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \ldots, \frac{\beta_r}{\alpha_r} | e\right)$,

$$
\det(K) = |H_1(\Sigma_2(S^3, K); \mathbb{Z})| = \left| \prod_{i=1}^{r} \alpha_i \left( e + \sum_{i=1}^{r} \frac{\beta_i}{\alpha_i} \right) \right|.
$$
For odd, fibered Montesinos knots,

\[ g(K) = \frac{1}{2} \left( \sum_{i=1}^{r} b^{(i)} + |e| - 1 \right) \]

We verify \( det(K) > 2g(K) + 1 \) for such knots.
For odd, fibered Montesinos knots,

\[ g(K) = \frac{1}{2} \left( \sum_{i=1}^{r} b^{(i)} + |e| - 1 \right) \]

We verify \( \det(K) > 2g(K) + 1 \) for such knots.

Finally, \( K = M\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{5} \right) \mid 1 \) is the knot 10_{145}. Since

\[ \Delta_{10_{145}}(t) = t^2 + t - 3 + t^{-1} + t^{-2}, \]

no odd fibered Montesinos knot admits an L-space surgery.
Even case

Similarly, pare down to the subfamily of fibered, even Montesinos knots which support the tight contact structure:

\[ M(\frac{-d_1}{2d_1+1}, \frac{-d_2}{2d_2+1}, \ldots, \frac{-d_r}{2d_r+1} | 1) \]

\[ M(\frac{-m_1}{m_1+1}, \frac{-m_2}{m_2+1}, \ldots, \frac{-m_r}{m_r+1} | 2) \]

**Lemma**

\[ M(\frac{-m_1}{m_1+1}, \ldots, \frac{-m_r}{m_r+1} | 2) \text{ are isotopic to pretzel links.} \]
Pretzel knots

Theorem (Lidman-M.)

*A pretzel knot admits an L-space surgery if and only if* \( K \cong T(2, 2n+1), \ n \geq 0, \) *or* \( K \cong \pm(-2, 3, 2n+1), \ n \geq 0. \)

- Gabai’s classification of fibered pretzel links.
- Determinant-genus inequality
- \( \Delta_K(t) \) obstructions using the Kauffman state sum:

\[
\Delta_K(T) = \sum_{x \in S} (-1)^{M(x)} T^{A(x)}
\]

*Figure:* Computations use existence of essential Conway spheres.
**Essential \( n \)-string tangle decompositions**

**Definition**

\( K \subset S^3 \) has an **essential \( n \)-string tangle decomposition** if \( \exists \) embedded sphere \( Q \) such that \( Q \cap K = \{2n \text{ pts}\} \) and where \( Q - \partial N(K) \) is essential in \( S^3 - N(K) \).

**Theorem (Krcatovich)**

*L-space knots are 1-string prime.*

**Conjecture (Lidman-M.)**

*L-space knots are 2-string prime.*

**Remark:** (Wu) \( \Rightarrow \) Amongst arborescent knots, a lens space knot cannot have an essential Conway sphere.
(Hayahsi-Matsuda-Ozawa): If a braided satellite knot has an essential tangle decomposition, then its companion has an essential tangle decomposition, too.

(Hom-Lidman-Vafaee): An L-space knot that is a Berge-Gabai satellite knot must have an L-space knot as its companion.

If there exists a Berge-Gabai L-space knot with an essential tangle decomposition, its companion will also be an L-space knot with an essential tangle decomposition.
What can we say about tunnel number?

- Many L-space knots have tunnel number one.
- Tunnel number one knots are \(n\)-string prime. (Gordon-Reid)
- For all \(N\), there exists an L-space knot with tunnel number \(N\). (Baker-M.)
- There exists a hyperbolic L-space knot with tunnel number two. (Motegi).
Thank you!