Homework 2
due October 16, 2001

(1) (a) Let $\sigma$ be the $m$-cycle $(a_1a_2\ldots a_m)$ in $S_n$. Show that $|\sigma| = m$.
(b) Show that the order of an element in $S_n$ is the least common multiple of the lengths of the cycles in its cycle decomposition.

(2) Let $\phi : G \to H$ be a homomorphism of groups, $A$ a subgroup of $G$, and $B$ a subgroup of $H$. Show that
(a) $\ker \phi$ and $\phi^{-1}(B) = \{a \in G \mid \phi(a) \in B\}$ are subgroups of $G$.
(b) $\phi(A)$ is a subgroup of $H$.

(3) Dummit, Foote I.1.7 Exercise 18 (page 45)
(4) Dummit, Foote I.1.7 Exercise 19 (page 45)

(5) Let $G$ and $H$ be groups. Define the direct product of $G$ and $H$ to be the set $G \times H$ with binary operation

$$(a, b)(a', b') = (aa', bb') \quad \text{where} \quad a, a' \in G \text{ and } b, b' \in H.$$ 

(a) Show that $G \times H$ is a group.
(b) Let $\langle a \rangle$ and $\langle b \rangle$ be finite cyclic groups of orders $m$ and $n$, respectively, which are relatively prime. Prove that $\langle a \rangle \times \langle b \rangle$ is cyclic.
(c) What about the converse?

(6) Dummit, Foote I.2.2 Exercise 10 (page 54)
(7) Dummit, Foote I.2.3 Exercise 26 (page 62)
Extra problem: Two elements $a, b \in G$ of the group $G$ are called
conjugate if there is a $c \in G$ such that $a = cbc^{-1}$.

(1) Prove that the cycles of maximal length in $S_n$ are conjugate.
How many cycles of maximal length are there in $S_n$?
(2) Consider the cycle $\xi = (12\cdots n) \in S_n$. What is the centralizer
of $\xi$?
(3) We showed in class that every permutation $\xi \in S_n$ can be writ-
ten as

$$\xi = \xi_1 \circ \cdots \circ \xi_r$$

where the $\xi_i$ are cycles and each number $1, 2, \ldots, n$ occurs in
precisely one cycle. Let $n_i$ denote the length of cycle $\xi_i$. We
may assume without loss of generality that

$$n_1 + \cdots + n_r = n$$

$$n_1 \geq n_2 \geq \cdots \geq n_r.$$ 

A tuple $(n_1, \ldots, n_r)$ with these properties is called a partition
of $n$. To each $\xi \in S_n$ we have associated a unique partition
of $n$. Prove the following: Two permutations $\xi_1, \xi_2 \in S_n$ are
conjugate if and only if the corresponding partitions of $n$ agree.