Homework Set Two: Fundamental Theorem of Algebra and Vector Spaces

Directions: Submit your solutions to the Calculational Exercises and the Proof-Writing Exercises separately at the beginning of lecture on Friday, October 12, 2007. The two problems sets will be graded by different persons.

Calculational Exercises

1. Given any complex number \( \alpha \in \mathbb{C} \), show that the coefficients of the polynomial
\[
(z - \alpha)(z - \overline{\alpha})
\]
are real numbers.

2. Show that the space \( V = \{(x_1, x_2, x_3) \in \mathbb{F}^3 \mid x_1 + 2x_2 + 2x_3 = 0\} \) forms a vector space.

3. Give an example of a nonempty subset \( U \subset \mathbb{R}^2 \) such that \( U \) is closed under scalar multiplication but is not a subspace of \( \mathbb{R}^2 \).

Proof-Writing Exercises

1. Let \( p(z) \) be a polynomial with real coefficients, and let \( \alpha \in \mathbb{C} \) be a complex number. Prove that \( p(\alpha) = 0 \) if and only \( p(\overline{\alpha}) = 0 \).

2. Let \( V \) be a vector space over \( \mathbb{F} \). Then, given \( a \in \mathbb{F} \) and \( v \in V \) such that \( av = 0 \), prove that either \( a = 0 \) or \( v = 0 \).

3. Let \( V \) be a vector space over \( \mathbb{F} \), and suppose that \( W_1 \) and \( W_2 \) are subspaces of \( V \). Prove that their intersection \( W_1 \cap W_2 \) is also a subspace of \( V \).