Homework Set Eight: Inner Product Spaces

Directions: Submit your solutions to the Calculational Exercises and the Proof-Writing Exercises separately at the beginning of lecture on Wednesday, November 28, 2007. The two problems sets will be graded by different persons.

Calculational Exercises

1. Let \((e_1, e_2, e_3)\) be the canonical basis of \(\mathbb{R}^3\), and define
   \[
   f_1 = e_1 + e_2 + e_3 \\
   f_2 = e_2 + e_3 \\
   f_3 = e_3.
   \]
   (a) Apply the Gram-Schmidt process to the basis \((f_1, f_2, f_3)\).
   (b) What do you obtain if you instead applied the Gram-Schmidt process to the basis \((f_3, f_2, f_1)\)?

2. Let \(\mathbb{R}_2[x]\) denote the inner product space of polynomials over \(\mathbb{R}\) having degree at most two, with inner product given by
   \[
   \langle f, g \rangle = \int_0^1 f(x)g(x)dx, \text{ for every } f, g \in \mathbb{R}_2[x].
   \]
   Apply the Gram-Schmidt procedure to the standard basis \(\{1, x, x^2\}\) for \(\mathbb{R}_2[x]\) in order to produce an orthonormal basis for \(\mathbb{R}_2[x]\).

Proof-Writing Exercises

1. Let \(V\) be a finite-dimensional inner product space over \(\mathbb{F}\), and let \(U\) be a subspace of \(V\). Prove that \(U = V\) if and only if the orthogonal complement \(U^\perp\) of \(U\) with respect to the inner product \(\langle \cdot, \cdot \rangle\) on \(V\) satisfies \(U^\perp = \{0\}\).

2. Let \(V\) be a finite-dimensional inner product space over \(\mathbb{F}\), and suppose that \(P \in \mathcal{L}(V)\) is a linear operator on \(V\) having the following two properties:
   (a) Given any vector \(v \in V\), \(P(P(v)) = P(v)\). I.e., \(P^2 = P\).
(b) Given any vector $u \in \text{null}(P)$ and any vector $v \in \text{range}(P)$, $\langle u, v \rangle = 0$.

Prove that $P$ is an orthogonal projection.