Homework Set Eight: Inner Product Spaces

Directions: Submit your solutions to the Calculational Exercises and the Proof-Writing Exercises **separately** at the **beginning** of lecture on **Wednesday, November 28, 2007**. The two problems sets will be graded by different persons.

Calculational Exercises

1. Let (e_1, e_2, e_3) be the canonical basis of $\mathbb{R}^3$, and define

$$\begin{aligned} f_1 &= e_1 + e_2 + e_3 \\ f_2 &= e_2 + e_3 \\ f_3 &= e_3. \end{aligned}$$

   (a) Apply the Gram-Schmidt process to the basis (f_1, f_2, f_3).

   (b) What do you obtain if you instead applied the Gram-Schmidt process to the basis (f_3, f_2, f_1)?

2. Let $\mathbb{R}_2[x]$ denote the inner product space of polynomials over $\mathbb{R}$ having degree at most two, with inner product given by

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx, \text{ for every } f, g \in \mathbb{R}_2[x].$$

   Apply the Gram-Schmidt procedure to the standard basis $\{1, x, x^2\}$ for $\mathbb{R}_2[x]$ in order to produce an orthonormal basis for $\mathbb{R}_2[x]$.

Proof-Writing Exercises

1. Let V be a finite-dimensional inner product space over $\mathbb{F}$, and let U be a subspace of V. Prove that $U = V$ if and only if the orthogonal complement $U^\perp$ of U with respect to the inner product $\langle \cdot , \cdot \rangle$ on V satisfies $U^\perp = \{0\}$.

2. Let V be a finite-dimensional inner product space over $\mathbb{F}$, and suppose that $P \in \mathcal{L}(V)$ is a linear operator on V having the following two properties:

   (a) Given any vector $v \in V$, $P(P(v)) = P(v)$. I.e., $P^2 = P$.

(b) Given any vector $u \in \text{null}(P)$ and any vector $v \in \text{range}(P)$, $\langle u, v \rangle = 0$.

Prove that P is an orthogonal projection.