Homework Set Nine: Orthogonality and Diagonalization

Directions: Submit your solutions to the Calculational Exercises and the Proof-Writing Exercises **separately** at the **beginning** of lecture on **Wednesday, December 5, 2007**. The two problems sets will be graded by different persons.

Calculational Exercises

1. Consider $\mathbb{R}^3$ with two orthonormal bases: the canonical basis $e = (e_1, e_2, e_3)$ and the basis $f = (f_1, f_2, f_3)$, where

$$f_1 = \frac{1}{\sqrt{3}}(1,1,1),\ f_2 = \frac{1}{\sqrt{6}}(1,-2,1),\ f_3 = \frac{1}{\sqrt{2}}(1,0,-1)\,.$$

   Find the matrix, S, of the change of basis transformation such that

$$[v]_f = S[v]_e, \quad \text{for all } v \in \mathbb{R}^3\,,$$

   where $[v]_b$ denotes the column vector of v with respect to the basis b.

2. Consider $\mathbb{R}^3$ with two orthonormal bases: the canonical basis $e = (e_1, e_2, e_3)$ and the basis $f = (f_1, f_2, f_3)$, where

$$f_1 = \frac{1}{\sqrt{3}}(1,1,1),\ f_2 = \frac{1}{\sqrt{6}}(1,-2,1),\ f_3 = \frac{1}{\sqrt{2}}(1,0,-1)\,.$$

   Find the canonical matrix, A, of the linear map $T \in \mathcal{L}(\mathbb{R}^3)$ with eigenvectors f_1, f_2, f_3 and eigenvalues $1, 1/2, -1/2$, respectively.

3. Let U be the subspace of $\mathbb{R}^3$ that coincides with the plane through the origin that is perpendicular to the vector $n = (1,1,1) \in \mathbb{R}^3$.

   (a) Find an orthonormal basis for U.

   (b) Find the matrix (with respect to the canonical basis on $\mathbb{R}^3$) of the orthogonal projection $P \in \mathcal{L}(\mathbb{R}^3)$ onto U, i.e., such that range$(P) = U$.

Proof-Writing Exercises

1. Let V be a finite-dimensional vector space over $\mathbb{F}$, and suppose that $T \in \mathcal{L}(V)$ satisfies $T^2 = T$. Prove that T is an orthogonal projection if and only if T is self-adjoint.

2. Let V be a finite-dimensional inner product space over $\mathbb{C}$, and suppose that $T \in \mathcal{L}(V)$ has the property that $T^* = -T$. (We call T a **skew Hermitian** operator on V.)

    (a) Prove that the operator $iT \in \mathcal{L}(V)$ defined by $(iT)(v) = i(T(v))$, for each $v \in V$, is Hermitian.

    (b) Prove that the canonical matrix for T can be unitarily diagonalized.

    (c) Prove that T has purely imaginary eigenvalues.