Problem 65 in Section 4.1 (Page 274)

Constructing a pipeline Supertankers off-load oil at a docking facility 4 mi offshore. The nearest refinery is 9 mi east of the shore point nearest the docking facility. A pipeline must be constructed connecting the docking facility with the refinery. The pipeline costs \$300,000 per mile if constructed underwater and \$200,000 per mile if overland.

a. Locate Point B to minimize the cost of the construction.

Solution: For this part and the next part, consider all pipe construction prices to be in units of \$100,000, i.e. the cost of pipeline underwater is then 3 such units. From the figure attached, we can construct the associated cost function, noting that the problem can be set as a function just in terms of how much land pipeline we can lay. If l represents the length of pipe on land, then $\sqrt{16+(9-l)^2}$ is the length of pipe underwater, giving us a cost function (in units of \$100,000)

$$C(l) = 3\sqrt{16+(9-l)^2} + 2l, \tag{1}$$

where $0 \le l \le 9$ is the domain (interval) of the cost function. To minimize the cost of construction, we want to find the critical l_c such that $\frac{dC}{dl}\Big|_{l_c} = 0$.

$$\frac{dC}{dl} = \frac{-3(9-l)}{\sqrt{16+(9-l)^2}} + 2 \tag{2}$$

So, we have then that

$$\begin{aligned} \frac{dC}{dl}\bigg|_{l_c} = 0 \quad &\Longrightarrow \quad \frac{2}{3} = \frac{9-l_c}{\sqrt{16+(9-l_c)^2}} \Longrightarrow 16+(9-l_c)^2 = \frac{9}{4}(9-l_c)^2 \\ &\Longrightarrow \quad 16 = \frac{5}{4}(9-l_c)^2 \Longrightarrow 9-l_c = \pm\frac{8}{\sqrt{5}} \Longrightarrow l_c = 9 \pm \frac{8}{\sqrt{5}}. \end{aligned}$$

But $l_c \in [0,9]$, so only $l_c = 9 - \frac{8}{\sqrt{5}}$ works. Plugging this value for l_c into (1) gives a cost of $C(l_c) = \$2.69$ million. Now to see if this is the minimum cost, we have to compare this to the endpoint cases, $l = 0$ and $l = 9$, corresponding to running pipeline directly underwater and going straight to the shore first, respectively. In each case, we have $C(0) = \$2.95$ million and $C(9) = \$3$ million. Thus, we see that our cost is minimized when Point B is 5.42 miles away from the refinery, or equivalently 3.58 miles away from Point A (as the back of the book has it).

b. The cost of underwater construction is expected to increase whereas the cost of overland construction is expected to stay constant. At what cost does it become optimal to construct the pipeline directly to Point A?

Solution: At first glance, and I was certainly guilty of this, one sees the word optimize and is inclined to start talking all new derivatives willy nilly. However, simply changing the original \$300,000 per mile of underwater pipeline to the variable p (in units of \$100,000) in (1), note that the problem is asking us to find the p such that $l_c = 9$ minimizes our cost function. This is equivalent to looking at what value l_c takes on for different p in (2), i.e. satisfying the minimization procedure. So starting with

$$\frac{dC}{dl}\bigg|_{l_c} = \frac{-p(9-l_c)}{\sqrt{16+(9-l_c)^2}} + 2 = 0 \tag{3}$$

we get after similar arithmetic manipulations as before the result that

$$l_c = 9 - \frac{8}{\sqrt{p^2-4}} \tag{4}$$

But note that in order for $l_c = 9$, this means in theory $p \to \infty$, which is realistically impossible. In other words, for however large the price of underwater pipe may be, there is always a point on the shore other than Point A that will minimize the construction cost. In fact, knowing that $l_c > 0$ and that $\sqrt{p^2 - 4} > 0$ you get the intersection of the two solution sets $p > 2$ and $p > \sqrt{388}/9$ indicating that for $p > \sqrt{388}/9 = \$218,864$ per mile of underwater pipe there will always be such a point on the shore that minimizes the cost for that p, and because we're told that the cost of the underwater pipe started at \$300,000 and is ever increasing, we are realistically guaranteed such a minimization point on the shore.

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