Homework Set Three: Linear Span and Bases

Directions: Submit your solutions to all homework problems **separately** at the **beginning** of lecture on **Friday, October 16, 2009**.

Calculational Exercises

1. Consider the real vector space $V = \mathbb{R}^4$. For each of the following five statements, provide either a proof or a counterexample.

   (a) $\dim V = 4$.

   (b) $\operatorname{span}((1,1,0,0),(0,1,1,0),(0,0,1,1)) = V$.

   (c) The list $((1,-1,0,0),(0,1,-1,0),(0,0,1,-1),(-1,0,0,1))$ is linearly independent.

   (d) Every list of four vectors $v_1, \ldots, v_4 \in V$, such that $\operatorname{span}(v_1, \ldots, v_4) = V$, is linearly independent.

   (e) Let v_1 and v_2 be two linearly independent vectors in V. Then, there exist vectors $u, w \in V$, such that (v_1, v_2, u, w) is a basis for V.

2. Consider the complex vector space $V = \mathbb{C}^3$ and the list (v_1, v_2, v_3) of vectors in V, where

$$v_1 = (i, 0, 0), \quad v_2 = (i, 1, 0), \quad v_3 = (i, i, -1).$$

   (a) Prove that $\operatorname{span}(v_1, v_2, v_3) = V$.

   (b) Prove or disprove: (v_1, v_2, v_3) is a basis for V.

Proof-Writing Exercises

1. Let V be a vector space over $\mathbb{F}$, and suppose that $(v_1, v_2, \ldots, v_n)$ is a linearly independent list of vectors in V. Given any $w \in V$ such that

$$(v_1 + w, v_2 + w, \ldots, v_n + w)$$

   is a linearly dependent list of vectors in V, prove that $w \in \operatorname{span}(v_1, v_2, \ldots, v_n)$.

2. Let $\mathbb{F}_m[z]$ denote the vector space of all polynomials with degree less than or equal to $m \in \mathbb{Z}_+$ and having coefficient over $\mathbb{F}$, and suppose that $p_0, p_1, \ldots, p_m \in \mathbb{F}_m[z]$ satisfy $p_j(2) = 0$. Prove that $(p_0, p_1, \ldots, p_m)$ is a linearly dependent list of vectors in $\mathbb{F}_m[z]$.

3. Let V be a finite-dimensional vector space over $\mathbb{F}$, and suppose that U is a subspace of V for which $\dim(U) = \dim(V)$. Prove that $U = V$.