Homework Set Four: Matrices and Linear Maps

Directions: Please submit your solutions to the Calculational Exercises and the Proof-Writing Exercises at the **beginning** of lecture on **Friday, October 23, 2009**.

Calculational Exercises

Submit solutions to Exercises 1, 2(i, m, r), 3(a), and 4(a).

1. Suppose that A, B, C, D, and E are matrices over $\mathbb{F}$ having the following sizes:

$$A \text{ is } 4 \times 5, \quad B \text{ is } 4 \times 5, \quad C \text{ is } 5 \times 2, \quad D \text{ is } 4 \times 2, \quad E \text{ is } 5 \times 4.$$

   Determine whether the following matrix expressions are defined, and, for those that are defined, determine the size of the resulting matrix.

   (a) BA (b) $AC + D$ (c) $AE + B$ (d) $AB + B$ (e) $E(A + B)$ (f) $E(AC)$

2. Suppose that A, B, C, D, and E are the following matrices:

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \text{ and } E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}.$$

   Determine whether the following matrix expressions are defined, and, for those that are defined, compute the resulting matrix.

   (a) $D + E$ (b) $D - E$ (c) $5A$ (d) $-7C$ (e) $2B - C$
   (f) $2E - 2D$ (g) $-3(D + 2E)$ (h) $A - A$ (i) AB (j) BA
   (k) $(3E)D$ (l) $(AB)C$ (m) $A(BC)$ (n) $(4B)C + 2B$ (o) $D - 3E$
   (p) $CA + 2E$ (q) $4E - D$ (r) DD

3. In each of the following, find matrices A, x, and b such that the given system of linear equations can be expressed as the single matrix equation $Ax = b$.

$$\text{(a)}\ \left.\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & 5x_3 & = & 7 \\ 9x_1 & - & x_2 & + & x_3 & = & -1 \\ x_1 & + & 5x_2 & + & 4x_3 & = & 0 \end{array}\right\} \qquad \text{(b)}\ \left.\begin{array}{rcrcrcrcr} 4x_1 & & & - & 3x_3 & + & x_4 & = & 1 \\ 5x_1 & + & x_2 & & & - & 8x_4 & = & 3 \\ 2x_1 & - & 5x_2 & + & 9x_3 & - & x_4 & = & 0 \\ & & 3x_2 & - & x_3 & + & 7x_4 & = & 2 \end{array}\right\}$$

4. In each of the following, express the matrix equation as a system of linear equations.

$$\text{(a)} \begin{bmatrix} 3 & -1 & 2 \\ 4 & 3 & 7 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 3 & -2 & 0 & 1 \\ 5 & 0 & 2 & -2 \\ 3 & 1 & 4 & 7 \\ -2 & 5 & 1 & 6 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Proof-Writing Exercises

1. Let U, V, and W be vector spaces over $\mathbb{F}$, and suppose that the linear maps $S \in \mathcal{L}(U, V)$ and $T \in \mathcal{L}(V, W)$ are both injective. Prove that the composition map $T \circ S$ is injective.

2. Let V and W be vector spaces over $\mathbb{F}$, and suppose that $T \in \mathcal{L}(V, W)$ is surjective. Given a spanning list $(v_1, \ldots, v_n)$ for V, prove that $\operatorname{span}(T(v_1), \ldots, T(v_n)) = W$.