Homework Set Five: Matrices and Linear Maps

Directions: Submit your solutions to the homework at the beginning of lecture on Friday, October 30, 2009.

Calculational Exercises

1. Define the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (x + y, x)$.
   
   (a) Show that $T$ is linear.
   
   (b) Show that $T$ is surjective.
   
   (c) Find $\dim(\ker(T))$.
   
   (d) Find the matrix for $T$ with respect to the canonical basis of $\mathbb{R}^2$.
   
   (e) Find the matrix for $T$ with respect to the canonical basis for the domain $\mathbb{R}^2$ and the basis $((1, 1), (1, -1))$ for the target space $\mathbb{R}^2$.
   
   (f) Show that the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) = (x + y, x + 1)$ is not linear.

2. Consider the complex vector spaces $\mathbb{C}^2$ and $\mathbb{C}^3$ with their canonical bases, and define $S \in \mathcal{L}(\mathbb{C}^3, \mathbb{C}^2)$ be the linear map defined by $S(v) = Av$, $\forall v \in \mathbb{C}^3$, where $A$ is the matrix

   $$ A = M(S) = \begin{pmatrix} i & 1 \\ 2i & -1 \end{pmatrix}. $$

   Find a basis for $\ker(S)$.

Proof-Writing Exercises

1. Let $V$ be a finite-dimensional vector space over $\mathbb{F}$ with $S, T \in \mathcal{L}(V, V)$. Prove that $T \circ S$ is invertible if and only if both $S$ and $T$ are invertible.