Homework Set Seven: Permutations and more on Eigenvalues

Directions: Submit your Homework at the **beginning** of lecture on **Friday, November 13, 2009**.

Calculational Exercises

1. Let $T \in \mathcal{L}(\mathbb{R}^2)$ be defined by
$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x+y \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 .$$
Define two real numbers λ_+ and λ_- as follows:
$$\lambda_+ = \frac{1+\sqrt{5}}{2}, \quad \lambda_- = \frac{1-\sqrt{5}}{2} .$$
   (a) Find the matrix of T with respect to the canonical basis for $\mathbb{R}^2$ (both as the domain and the codomain of T; call this matrix A).
   (b) Verify that λ_+ and λ_- are eigenvalues of T by showing that v_+ and v_- are eigenvectors, where
$$v_+ = \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix}, \quad v_- = \begin{pmatrix} 1 \\ \lambda_- \end{pmatrix} .$$
   (c) Show that (v_+, v_-) is a basis of $\mathbb{R}^2$.
   (d) Find the matrix of T with respect to the basis (v_+, v_-) for $\mathbb{R}^2$ (both as the domain and the codomain of T; call this matrix B).

2. (a) For each permutation $\pi \in \mathcal{S}_3$, compute the number of inversions in π, and classify π as being either an even or an odd permutation.
   (b) Use your result from Part (a) to construct a formula for the determinant of a 3×3 matrix.

3. Let $A \in \mathbb{C}^{3\times 3}$ be given by
$$A = \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & -1 \end{bmatrix} .$$
   (a) Calculate $\det(A)$.
   (b) Find $\det(A^4)$.

Proof-Writing Exercises

1. (a) Let $a, b, c, d \in \mathbb{F}$ and consider the system of equations given by

$$ax_1 + bx_2 = 0 \tag{1}$$
$$cx_1 + dx_2 = 0. \tag{2}$$

Note that $x_1 = x_2 = 0$ is a solution for any choice of a, b, c, and d. Prove that this system of equations has a non-trivial solution if and only if $ad - bc = 0$.

(b) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{F}^{2\times 2}$, and recall that we can define a linear operator $T \in \mathcal{L}(\mathbb{F}^2)$ on $\mathbb{F}^2$ by setting $T(v) = Av$ for each $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{F}^2$.

Show that the eigenvalues for T are exactly the $\lambda \in \mathbb{F}$ for which $p(\lambda) = 0$, where $p(z) = (a - z)(d - z) - bc$.

Hint: Write the eigenvalue equation $Av = \lambda v$ as $(A - \lambda I)v = 0$ and use the first part.

2. Let V be a finite-dimensional vector space over $\mathbb{F}$, and let $S, T \in \mathcal{L}(V)$ be linear operators on V. Suppose that T has $\dim(V)$ distinct eigenvalues and that, given any eigenvector $v \in V$ for T associated to some eigenvalue $\lambda \in \mathbb{F}$, v is also an eigenvector for S associated to some (possibly distinct) eigenvalue $\mu \in \mathbb{F}$. Prove that $T \circ S = S \circ T$.