Homework Set Seven: Permutations and more on Eigenvalues

Directions: Submit your Homework at the beginning of lecture on Friday, November 13, 2009.

Calculational Exercises

1. Let $T \in \mathcal{L}(\mathbb{R}^2)$ be defined by
   \[
   T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x + y \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.
   \]

   Define two real numbers $\lambda_+$ and $\lambda_-$ as follows:
   \[
   \lambda_+ = \frac{1 + \sqrt{5}}{2}, \quad \lambda_- = \frac{1 - \sqrt{5}}{2}.
   \]

   (a) Find the matrix of $T$ with respect to the canonical basis for $\mathbb{R}^2$ (both as the domain and the codomain of $T$; call this matrix $A$).

   (b) Verify that $\lambda_+$ and $\lambda_-$ are eigenvalues of $T$ by showing that $v_+$ and $v_-$ are eigenvectors, where

   \[
   v_+ = \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix}, \quad v_- = \begin{pmatrix} 1 \\ \lambda_- \end{pmatrix}.
   \]

   (c) Show that $(v_+, v_-)$ is a basis of $\mathbb{R}^2$.

   (d) Find the matrix of $T$ with respect to the basis $(v_+, v_-)$ for $\mathbb{R}^2$ (both as the domain and the codomain of $T$; call this matrix $B$).

2. (a) For each permutation $\pi \in S_3$, compute the number of inversions in $\pi$, and classify $\pi$ as being either an even or an odd permutation.

   (b) Use your result from Part (a) to construct a formula for the determinant of a $3 \times 3$ matrix.

3. Let $A \in \mathbb{C}^{3 \times 3}$ be given by

   \[
   A = \begin{bmatrix}
   1 & 0 & i \\
   0 & 1 & 0 \\
   -i & 0 & -1
   \end{bmatrix}.
   \]

   (a) Calculate $\text{det}(A)$.

   (b) Find $\text{det}(A^4)$.  
Proof-Writing Exercises

1. (a) Let \(a, b, c, d \in \mathbb{F}\) and consider the system of equations given by

\[
ax_1 + bx_2 = 0 \quad (1) \\

\]

\[
Cx_1 + dx_2 = 0. \quad (2)
\]

Note that \(x_1 = x_2 = 0\) is a solution for any choice of \(a, b, c,\) and \(d\). Prove that this system of equations has a non-trivial solution if and only if \(ad - bc = 0\).

(b) Let \(A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{F}^{2 \times 2}\), and recall that we can define a linear operator \(T \in \mathcal{L}(\mathbb{F}^2)\) on \(\mathbb{F}^2\) by setting \(T(v) = Av\) for each \(v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{F}^2\).

Show that the eigenvalues for \(T\) are exactly the \(\lambda \in \mathbb{F}\) for which \(p(\lambda) = 0\), where \(p(z) = (a - z)(d - z) - bc\).

**Hint:** Write the eigenvalue equation \(Av = \lambda v\) as \((A - \lambda I)v = 0\) and use the first part.

2. Let \(V\) be a finite-dimensional vector space over \(\mathbb{F}\), and let \(S, T \in \mathcal{L}(V)\) be linear operators on \(V\). Suppose that \(T\) has \(\dim(V)\) distinct eigenvalues and that, given any eigenvector \(v \in V\) for \(T\) associated to some eigenvalue \(\lambda \in \mathbb{F}\), \(v\) is also an eigenvector for \(S\) associated to some (possibly distinct) eigenvalue \(\mu \in \mathbb{F}\). Prove that \(T \circ S = S \circ T\).