

Homework Set Eight: Inner Product Spaces

Directions: Please submit your solutions at the **beginning** of lecture on **Wednesday, November 25, 2009**.

Computational Exercises

1. Let (e_1, e_2, e_3) be the canonical basis of \mathbb{R}^3 , and define

$$f_1 = e_1 + e_2 + e_3$$

$$f_2 = e_2 + e_3$$

$$f_3 = e_3.$$

- (a) Apply the Gram-Schmidt process to the basis (f_1, f_2, f_3) .
(b) What do you obtain if you instead applied the Gram-Schmidt process to the basis (f_3, f_2, f_1) ?
2. Let $\mathbb{R}_2[x]$ denote the inner product space of polynomials over \mathbb{R} having degree at most two, with inner product given by

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx, \text{ for every } f, g \in \mathbb{R}_2[x].$$

Apply the Gram-Schmidt procedure to the standard basis $\{1, x, x^2\}$ for $\mathbb{R}_2[x]$ in order to produce an orthonormal basis for $\mathbb{R}_2[x]$.

Proof-Writing Exercises

1. Let V be a finite-dimensional inner product space over \mathbb{F} . Given any vectors $u, v \in V$, prove that the following two statements are equivalent:
- (a) $\langle u, v \rangle = 0$
(b) $\|u\| \leq \|u + \alpha v\|$ for every $\alpha \in \mathbb{F}$.
2. Let V be a finite-dimensional inner product space over \mathbb{F} , and let U be a subspace of V . Prove that $U = V$ if and only if the orthogonal complement U^\perp of U with respect to the inner product $\langle \cdot, \cdot \rangle$ on V satisfies $U^\perp = \{0\}$.

3. Let V be a finite-dimensional inner product space over \mathbb{F} , and suppose that $P \in \mathcal{L}(V)$ is a linear operator on V having the following two properties:

(a) Given any vector $v \in V$, $P(P(v)) = P(v)$. I.e., $P^2 = P$.

(b) Given any vector $u \in \text{null}(P)$ and any vector $v \in \text{range}(P)$, $\langle u, v \rangle = 0$.

Prove that P is an orthogonal projection.