Practice Problems

The Final Exam will consist of 8 questions, one of which is a question where you will need to state definitions or theorems from class, and one of which is a true or false question. (You do not need to know how to do Problem 6(c) as we did not cover that!).

1. Let \( v_1, v_2, v_3 \in \mathbb{R}^3 \) be given by \( v_1 = (1, 2, 1), \ v_2 = (1, -2, 1), \) and \( v_3 = (1, 2, -1). \) Apply the Gram-Schmidt procedure to the basis \( (v_1, v_2, v_3) \) of \( \mathbb{R}^3, \) and call the resulting orthonormal basis \( (u_1, u_2, u_3). \)

2. Let \( A \in \mathbb{C}^{3 \times 3} \) be given by
\[
A = \begin{bmatrix}
1 & 0 & i \\
0 & 1 & 0 \\
-i & 0 & -1
\end{bmatrix}.
\]

(a) Calculate \( \det(A). \)
(b) Find \( \det(A^4). \)

3. Let \( P \subset \mathbb{R}^3 \) be the plane containing 0 perpendicular to the vector \( (1, 1, 1). \) Using the standard norm, calculate the distance of the point \( (1, 2, 3) \) to \( P. \)

4. Let \( V = \mathbb{C}^4 \) with its standard inner product. For \( \theta \in \mathbb{R}, \) let
\[
v_\theta = \begin{pmatrix}
1 \\
e^{i\theta} \\
e^{2i\theta} \\
e^{3i\theta}
\end{pmatrix} \in \mathbb{C}^4.
\]

Find the canonical matrix of the orthogonal projection onto the subspace \( \{v_\theta\}^\perp. \)

5. Let \( r \in \mathbb{R} \) and let \( T \in \mathcal{L}(\mathbb{C}^2) \) be the linear map with canonical matrix
\[
T = \begin{pmatrix}
1 & -1 \\
-1 & r
\end{pmatrix}.
\]

(a) Find the eigenvalues of \( T. \)
(b) Find an orthonormal basis of \( \mathbb{C}^2 \) consisting of eigenvectors of \( T. \)
(c) Find a unitary matrix \( U \) such that \( UTU^* \) is diagonal.
6. Let $A$ be the complex matrix given by:

$$A = \begin{bmatrix}
5 & 0 & 0 \\
0 & -1 & -1 + i \\
0 & -1 - i & 0
\end{bmatrix}$$

(a) Find the eigenvalues of $A$.
(b) Find an orthonormal basis of eigenvectors of $A$.
(c) Calculate $|A| = \sqrt{A^*A}$.
(d) Calculate $e^A$.

7. Give an orthonormal basis for $\text{null}(T)$, where $T \in \mathcal{L}(\mathbb{C}^4)$ is the map with canonical matrix

$$
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix}.
$$

8. Describe the set of solutions $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ of the system of equations

$$
\begin{align*}
x_1 - x_2 + x_3 &= 0 \\
x_1 + 2x_2 + x_3 &= 0 \\
2x_1 + x_2 + 2x_3 &= 0
\end{align*}
$$

9. Prove or give a counterexample: For any $n \geq 1$ and $A, B \in \mathbb{R}^{n \times n}$, one has

$$\det(A + B) = \det(A) + \det(B).$$

10. Prove or give a counterexample: For any $r \in \mathbb{R}$, $n \geq 1$ and $A \in \mathbb{R}^{n \times n}$, one has

$$\det(rA) = r \det(A).$$

11. Prove or give a counterexample: For any $n \geq 1$ and $A \in \mathbb{C}^{n \times n}$, one has

$$\text{null}(A) = (\text{range}(A))^\perp.$$

12. Prove or give a counterexample: The Gram-Schmidt process applied to an orthonormal list of vectors reproduces that list unchanged.

13. Prove or give a counterexample: Every unitary matrix is invertible.