LECTURE 5: WEAK TABLEAUX

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1. Weak Horizontal Strips

Definition 1.1. Let τ and κ be $(k+1)$-cores. We say κ/τ is a *weak horizontal strip* of size r if κ/τ is a horizontal strip and there exists a saturated chain of length r

$$\tau \to_k \tau^{(1)} \to_k \tau^{(2)} \to_k \cdots \to_k \tau^{(r)} = \kappa.$$

Proposition 1.2 (k-bounded characterization)**.** *Let $\tau \subseteq \kappa$ be $(k+1)$-cores. Then κ/τ is a weak horizontal strip if and only if $P(\kappa)/P(\tau)$ is a horizontal strip and $P(\kappa^t)/P(\tau^t)$ is a vertical strip.*

Definition 1.3. An element $w \in \widetilde{S}_n$ is *cyclically decreasing* if $w = s_{i_1} \cdots s_{i_l}$ with no index repeated and j precedes $j-1$ modulo n when both are in the set $\{i_1, \ldots, i_l\}$.

Example 1.4. Let $n = 6$. Then $w = s_3 s_2 s_0 s_5 = s_0 s_5 s_3 s_2$ is cyclically decreasing, however $s_1 s_2$ is not. In particular, we cannot use all generators such as $s_5 s_4 s_3 s_2 s_1 s_0$ since 0 does not precede 5 (recall modulo 6).

Proposition 1.5. *Let $\tau \subseteq \kappa$ be $(k+1)$-cores. Then κ/τ is a weak horizontal strip if and only if $\kappa = w\tau$ where w is a cyclically decreasing element.*

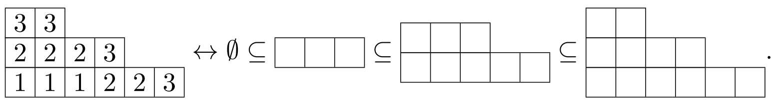
Proof sketch. Take a core and add boxes of s_{i_j}. If this would be an element breaking the conditions of cyclically decreasing, then it would correspond to adding a box on top of a previously added box. Thus κ/τ would not be a horizontal strip, noting that increasing in the weak order always corresponds to adding boxes. □

2. Pieri Rule to Tableaux

Recall that $h_\mu = h_{\mu_1} \cdots h_{\mu_d} s_\emptyset$ and the Pieri rule is $h_r s_\lambda = \sum_\mu s_\mu$ where the sum was over all partitions μ such that μ/λ is a horizontal r-strip. Now if we iteratively apply the Pieri rule, we note that we are building a semi-standard Young tableaux. Alternatively by the column strict condition, every semi-standard Young tableau of shape λ and weight μ can be thought of a sequence of partitions $\left(\lambda^{(i)}\right)_i$ such that

$$\emptyset \subseteq \lambda^{(1)} \subseteq \lambda^{(2)} \subseteq \cdots \subseteq \lambda^{(d)} = \lambda$$

where $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal μ_i-strip. For example, consider



Now if we look all such semi-standard Young tableaux of a given shape $\lambda \vdash |\mu|$, we note that we get fillings of weight μ. Therefore we can express $h_\mu = \sum_{\lambda \vdash |\mu|} K_{\lambda\mu} s_\lambda$ recalling $K_{\lambda\mu}$ is called a Kostka number and is the number of semi-standard Young

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tableaux of shape λ and weight (content) μ. Recall that $K_{\lambda\mu} = 0$ unless $\mu \trianglelefteq \lambda$ (λ dominates μ or λ is greater than μ in the dominance order) and $K_{\lambda\lambda} = 1$.

Therefore if we consider the matrix $(K_{\lambda\mu})_{\lambda\mu}$, it is invertible (as a matrix). Also recall for the Hall inner product, we have

$$\langle h_\lambda, m_\lambda \rangle = \delta_{\lambda\mu} = \langle s_\lambda, s_\mu \rangle.$$

Hence the Pieri rule defines s_λ since $\langle h_\mu, s_\lambda \rangle = K_{\lambda\mu}$. This also implies that

$$s_\lambda = \sum_\mu \langle s_\lambda, h_\mu \rangle m_\mu = \sum_{\mu \vdash |\lambda|} K_{\lambda\mu} m_\mu.$$

3. Weak Tableaux

Recall that $\Lambda_{(k)} = \mathbb{Q}[h_1, \ldots, h_k]$ and $\Lambda^{(k)} = \Lambda / \langle m_\lambda \mid \lambda_1 > k \rangle$ and the k-Pieri rule is $h_r s_\mu^{(k)} = \sum_\lambda s_\lambda^{(k)}$ where we sum over all λ such that λ/μ is a weak horizontal r-strip.

Example 3.1. Let $k = 4$ and consider $h_{431} = h_1 h_3 h_4 s_\emptyset^{(4)}$. Thus we have

$$h_1 h_3 s^{(4)}_{(4)} = h_1 s^{(4)}_{(7,3)} = s^{(4)}_{(7,3,1)} + s^{(4)}_{(8,4)}$$

where the first equality corresponds to multiplying by $s_3 s_2 s_1 s_0$, the second by $s_1 s_0 s_4$, and the last by s_3 or s_2 (hence two terms). In terms of tableaux, we have

$$\begin{array}{|c|c|c|c|c|}\hline 0&1&2&3&4\\\hline\end{array} \subseteq \begin{array}{|c|c|c|c|c|}\hline 4&0&1\\\hline 0&1&2&3&4\\\hline\end{array} \subseteq \begin{array}{|c|c|c|c|c|}\hline 3\\\hline 4&0&1&2\\\hline 0&1&2&3&4\\\hline\end{array}$$

where the entries are the residue and the last one we added either the 3 or the 2.

We now want an analogous definition of k-Schur functions in terms of monomial symmetric functions, so we need the notion of a weak tableau.

Definition 3.2. A *weak tableau* is a sequence of $(k+1)$-cores $\emptyset \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(d)} = \lambda$ such that $\lambda^{(i)}/\lambda^{(i-1)}$ is a weak horizontal strip. We say the same is λ and the weight (content) is α where $\alpha_i = |\lambda^{(i)}/\lambda^{(i-1)}|_{k+1}$.

Remark 3.3. We note that α_i does not record the number of i's in the tableaux. Instead it records the number of distinct residues appearing in $\lambda^{(i)}/\lambda^{(i-1)}$.

Example 3.4. For h_{431} we had the tableaux

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 3\\\hline 2&2&2\\\hline 1&1&1&1&2&2&2\\\hline\end{array} \xrightarrow{P} \begin{array}{|c|c|c|c|}\hline 3\\\hline 2&2&2\\\hline 1&1&1&1\\\hline\end{array}$$

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 2&2&2&3\\\hline 1&1&1&1&2&2&2&3\\\hline\end{array} \xrightarrow{P} \begin{array}{|c|c|c|c|}\hline 2&2&2&3\\\hline 1&1&1&1\\\hline\end{array}$$

Hence if λ is a $(k+1)$-core and α a tuple of non-negative integers such that $\sum_i \alpha_i = |\lambda|_{k+1}$, a weak tableaux of weight α is a semi-standard filling of shape λ with letters $1, 2, \ldots, d$ such that the collection of cells filled with i occupies α_i distinct $k+1$ residues.

Let $K^{(k)}_{\lambda\mu}$ denote the number of weak tableaux of $(k+1)$-core shape λ and k-bounded weight μ, and call $K^{(k)}_{\lambda\mu}$ the k-Kostka numbers. In particular $K^{(k)}_{\lambda\mu} = 1$ if $P(\lambda) = \mu$ and $K^{(k)}_{\lambda\mu} = 0$ if $P(\lambda) \trianglelefteq \mu$. Thus we mostly have an analog of the results of Schur functions in that $h_\mu = \sum_\lambda K^{(k)}_{\lambda\mu} s^{(k)}_\lambda$ and $s^{(k)}_\lambda$ is a well-defined basis for $\Lambda_{(k)}$ since the the k-Kostka matrix is invertible. However k-Schur functions no longer pair with themselves, so this leads us into dual k-Schur functions.

Using the action of s_i on $(k+1)$-cores, we would add all boxes of residue i translates to k-bounded partitions by adding the top most box of residue i. Thus we could work with k-bounded partitions and our usual notion of weight is preserved.