LECTURE 5: WEAK TABLEAUX

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1. WEAK HORIZONTAL STRIPS

Definition 1.1. Let \( \tau \) and \( \kappa \) be \((k + 1)\)-cores. We say \( \kappa/\tau \) is a weak horizontal strip of size \( r \) if \( \kappa/\tau \) is a horizontal strip and there exists a saturated chain of length \( r \)

\[
\tau \to_k \tau(1) \to_k \tau(2) \to_k \cdots \to_k \tau(r) = \kappa.
\]

Proposition 1.2 \((k\text{-bounded characterization})\). Let \( \tau \subseteq \kappa \) be \((k + 1)\)-cores. Then \( \kappa/\tau \) is a weak horizontal strip if and only if \( P(\kappa)/P(\tau) \) is a horizontal strip and \( P(\kappa^i)/P(\tau^i) \) is a vertical strip.

Definition 1.3. An element \( w \in \tilde{S}_n \) is cyclically decreasing if \( w = s_{i_1} \cdots s_{i_l} \) with no index repeated and \( j \) precedes \( j - 1 \) modulo \( n \) when both are in the set \( \{i_1, \ldots, i_l\} \).

Example 1.4. Let \( n = 6 \). Then \( w = s_3 s_2 s_0 s_5 = s_0 s_5 s_3 s_2 \) is cyclically decreasing, however \( s_1 s_2 \) is not. In particular, we cannot use all generators such as \( s_5 s_4 s_3 s_2 s_1 s_0 \) since 0 does not precede 5 (recall modulo 6).

Proposition 1.5. Let \( \tau \subseteq \kappa \) be \((k + 1)\)-cores. Then \( \kappa/\tau \) is a weak horizontal strip if and only if \( \kappa = w \tau \) where \( w \) is a cyclically decreasing element.

Proof sketch. Take a core and add boxes of \( s_{i_j} \). If this would be an element breaking the conditions of cyclically decreasing, then it would correspond to adding a box on top of a previously added box. Thus \( \kappa/\tau \) would not be a horizontal strip, noting that increasing in the weak order always corresponds to adding boxes. \( \square \)

2. PIERI RULE TO TABLEAUX

Recall that \( h_\mu = h_{\mu_1} \cdots h_{\mu_d} s_\emptyset \) and the Pieri rule is \( h_r s_\lambda = \sum \mu \) where the sum was over all partitions \( \mu \) such that \( \mu/\lambda \) is a horizontal \( r \)-strip. Now if we iteratively apply the Pieri rule, we note that we are building a semi-standard Young tableaux. Alternatively by the column strict condition, every semi-standard Young tableau of shape \( \lambda \) and weight \( \mu \) can be thought of a sequence of partitions \( (\lambda^{(i)})_i \) such that

\[
\emptyset \subseteq \lambda^{(1)} \subseteq \lambda^{(2)} \subseteq \cdots \subseteq \lambda^{(d)} = \lambda
\]

where \( \lambda^{(i)}/\lambda^{(i-1)} \) is a horizontal \( \mu_i \)-strip. For example, consider

\[
\begin{array}{ccc}
3 & 3 \\
2 & 2 & 2 & 3 \\
1 & 1 & 1 & 2 & 2 & 3
\end{array}
\]

Now if we look all such semi-standard Young tableaux of a given shape \( \lambda \vdash |\mu| \), we note that we get fillings of weight \( \mu \). Therefore we can express \( h_\mu = \sum_{\lambda \vdash |\mu|} K_{\lambda \mu} s_\lambda \)

\( \text{recalling } K_{\lambda \mu} \text{ is called a Kostka number and is the number of semi-standard Young} \)

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tableaux of shape $\lambda$ and weight (content) $\mu$. Recall that $K_{\lambda\mu} = 0$ unless $\mu \subseteq \lambda$ (\(\lambda\) dominates $\mu$ or $\lambda$ is greater than $\mu$ in the dominance order) and $K_{\lambda\lambda} = 1$.

Therefore if we consider the matrix $(K_{\lambda\mu})_{\lambda\mu}$, it is invertible (as a matrix). Also recall for the Hall inner product, we have

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu} = \langle s_\lambda, s_\mu \rangle.$$

Hence the Pieri rule defines $s_\lambda$ since $\langle h_\mu, s_\lambda \rangle = K_{\lambda\mu}$. This also implies that

$$s_\lambda = \sum_{\mu} \langle s_\lambda, h_\mu \rangle m_\mu = \sum_{\mu \vdash \lambda} K_{\lambda\mu} m_\mu.$$

### 3. Weak Tableaux

Recall that $\Lambda^{(k)} = \mathbb{Q}[h_1, \ldots, h_k]$ and $\Lambda^{(k)} = \Lambda / (m_\lambda | \lambda_1 > k)$ and the $k$-Pieri rule is $h_r s^{(k)}_\mu = \sum \lambda s^{(k)}_\lambda$ where we sum over all $\lambda$ such that $\lambda / \mu$ is a weak horizontal strip.

**Example 3.1.** Let $k = 4$ and consider $h_{431} = h_1 h_3 h_4 s^{(4)}_\emptyset$. Thus we have

$$h_1 h_3 s^{(4)}_\emptyset = h_1^{(4)} = s^{(4)}_1 + s^{(4)}_2,$$

where the first equality corresponds to multiplying by $s_3 s_2 s_1 s_0$, the second by $s_1 s_0 s_4$, and the last by $s_3$ or $s_2$ (hence two terms). In terms of tableaux, we have

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
0 & 1 & 2 & 3 \\
0 & 1 & 2 & 3 \\
0 & 1 & 2 & 3 \\
\end{array} \quad \subseteq \quad \begin{array}{cccc}
4 & 0 & 1 & 0 \\
0 & 1 & 2 & 3 \\
0 & 1 & 2 & 3 \\
0 & 1 & 2 & 3 \\
\end{array} \quad \subseteq \quad \begin{array}{cccc}
3 & 4 & 0 & 1 \\
4 & 0 & 1 & 2 \\
0 & 1 & 2 & 3 \\
0 & 1 & 2 & 3 \\
\end{array}
\]

where the entries are the residue and the last one we added either the 3 or the 2.

We now want an analogous definition of $k$-Schur functions in terms of monomial symmetric functions, so we need the notion of a weak tableau.

**Definition 3.2.** A weak tableau is a sequence of $(k+1)$-cores $\emptyset \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(d)} = \lambda$ such that $\lambda^{(i)}/\lambda^{(i-1)}$ is a weak horizontal strip. We say the same is $\lambda$ and the weight (content) is $\alpha$ where $\alpha_i = |\lambda^{(i)}/\lambda^{(i-1)}|_{k+1}$.

**Remark 3.3.** We note that $\alpha_i$ does not record the number of $i$’s in the tableaux. Instead it records the number of distinct residues appearing in $\lambda^{(i)}/\lambda^{(i-1)}$.

**Example 3.4.** For $h_{431}$ we had the tableaux

\[
\begin{array}{cccc}
3 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 \\
\end{array} \quad \rightarrow \quad \begin{array}{cccc}
3 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccc}
2 & 2 & 2 & 3 \\
1 & 1 & 1 & 1 \\
\end{array} \quad \rightarrow \quad \begin{array}{cccc}
2 & 2 & 2 & 3 \\
1 & 1 & 1 & 1 \\
\end{array}
\]

Hence if $\lambda$ is a $(k+1)$-core and $\alpha$ a tuple of non-negative integers such that $\sum \alpha_i = |\lambda|_{k+1}$, a weak tableaux of weight $\alpha$ is a semi-standard filling of shape $\lambda$ with letters $1, 2, \ldots, d$ such that the collection of cells filled with $i$ occupies $\alpha_i$ distinct $k+1$ residues.
Let $K^{(k)}_{\lambda \mu}$ denote the number of weak tableaux of $(k + 1)$-core shape $\lambda$ and $k$-bounded weight $\mu$, and call $K^{(k)}_{\lambda \mu}$ the $k$-Kostka numbers. In particular $K^{(k)}_{\lambda \mu} = 1$ if $P(\lambda) = \mu$ and $K^{(k)}_{\lambda \mu} = 0$ if $P(\lambda) \sqsubseteq \mu$. Thus we mostly have an analog of the results of Schur functions in that $h^{(k)}_{\mu} = \sum_{\lambda} K^{(k)}_{\lambda \mu} s^{(k)}_{\lambda}$ and $s^{(k)}_{\lambda}$ is a well-defined basis for $\Lambda_{(k)}$ since the $k$-Kostka matrix is invertible. However $k$-Schur functions no longer pair with themselves, so this leads us into dual $k$-Schur functions.

Using the action of $s_i$ on $(k + 1)$-cores, we would add all boxes of residue $i$ translates to $k$-bounded partitions by adding the top most box of residue $i$. Thus we could work with $k$-bounded partitions and our usual notion of weight is preserved.