LECTURE 7: K-SCHUR FUNCTIONS IN THE NILCOXETER ALGEBRA

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We recall the following definitions.

Definition 0.1. A k-Schur function is defined as $s_\lambda^{(k)} = \sum_{\mu:\mu_1\le k} \gamma_{\lambda\mu} h_\mu$.

Definition 0.2. A noncommutative k-Schur function is $\not{s}_\lambda^{(k)} = \sum_{\mu:\mu_1\le k} \gamma_{\lambda\mu} \not{h}_\mu$, where

$\not{h}_r = \sum_{J\subset I, |J|=r} A_J^{dec}$ and $\not{h}_\mu = \not{h}_{\mu_1}\cdots \not{h}_{\mu_m}$.

Definition 0.3. The affine Stanley symmetric functions are $F_w = \sum_{\alpha\models n} \langle A_w, \not{h}_{\alpha_1}\not{h}_{\alpha_2}\cdots\rangle x^\alpha$ for $w\in \tilde{S}_n$ and $\langle A_w, A_v\rangle = \delta_{w,v}$.

If $w \in \tilde{S}_n/S_n$ (w an affine Grassmannian element), then $F_\lambda = \sigma_\lambda^{(k)}$ the dual k-Schur functions.

1. Cauchy Identity

Λ denotes the ring of symmetric functions. $h_r = \sum_{\lambda\vdash r} m_\lambda = \sum_{i_1\le i_2\le\cdots\le i_r} x_{i_1}x_{i_2}\cdots x_{i_r}$. $h_0 = m_\emptyset = 1$, $h_\lambda = h_{\lambda_1}h_{\lambda_2}\cdots$.

Proposition 1.1. *Let $\lambda\vdash r$ and $\alpha = (\alpha_1,\alpha_2,\cdots)$ a weak composition of r (zeros are allowed). Then the coefficient $N_{\lambda\alpha}$ of x^α in h_λ ($h_\lambda = \sum_{\mu\vdash r} N_{\lambda\mu} m_\mu$) is the number of $\mathbb{Z}_{\le 0}$ matrices $A = (a_{ij})_{i,j\ge 1}$ such that row(A)=λ and col(A)=α.*

Proof. The term x^α in h_λ is obtained by choosing $x_1^{a_{i_1}}x_2^{a_{i_2}}\cdots$ from each h_{λ_i} such that $\prod_i x_1^{a_{i_1}}x_2^{a_{i_2}}\cdots = x^\alpha$. This is the same as choosing a matrix (a_{ij}) with row(A)=λ and col(A)=α. □

Proposition 1.2. $\prod_{i,j\ge 1}\frac{1}{1-x_iy_j} = \sum_{\lambda,\mu\in P} N_{\lambda\mu} m_\lambda(x) m_\mu(y) = \sum_{\lambda\in P} m_\lambda(x) h_\lambda(y)$.

Proof. The monomial $x^\alpha y^\beta$ appearing in $\prod_{i,j\ge 1}\frac{1}{1-x_iy_j}$ corresponds to a non-negative integer matrix $A = (a_{ij})$ such that $\prod_{i,j\ge 1}(x_iy_j)^{a_{ij}} = x^\alpha y^\beta$, hence it is $N_{\lambda\mu}$. □

Date: October 22, 2012.

Definition 1.3. A pair of bases $\{u_\lambda\}, \{v_\lambda\}$ of Λ are dual if $\langle u_\lambda, v_\lambda\rangle = \delta_{\lambda\mu}$.

Proposition 1.4. *If $\{u_\lambda | \lambda \vdash r\}$ and $\{v_\lambda | \lambda \vdash r\}$ are bases of Λ^r (graded piece of degree r), then $\{u_\lambda | \lambda \vdash r\}$ and $\{v_\lambda | \lambda \vdash r\}$ are dual bases iff*

$$\sum_{\lambda \in P} u_\lambda(x) v_\lambda(y) = \prod_{i,j\geq 1} \frac{1}{1 - x_i y_j} = \sum_{\lambda\in P} m_\lambda(x) h_\lambda(y). \tag{1.1}$$

Proof. Write $m_\lambda = \sum_\rho \zeta_{\lambda_\rho} u_\rho$, $h_\mu = \sum_\nu \eta_{\mu_\nu} v_\nu$. Then

$$\delta_{\lambda\mu} = \langle m_\lambda, h_\mu\rangle = \sum_{\rho,\nu} \zeta_{\lambda_\rho}\eta_{\mu_\nu}\langle u_\rho, v_\nu\rangle = 1.1.$$

For fixed r ζ, η are matrices indexed by P_r (partitions of size r). Let $A_{\rho\nu} = \langle u_\rho, v_\nu\rangle$. Hence $1.1 \leftrightarrow I = \zeta A \eta^t$. Therefore $\{u_\lambda | \lambda \vdash r\}$ and $\{v_\lambda | \lambda \vdash r\}$ are dual iff $A = I$ and by 1.1 this is iff $I = \zeta\eta^t \leftrightarrow I = \zeta^t \eta \leftrightarrow \delta_{\rho\nu} = \sum_\lambda m_\lambda(x) h_\lambda(y)$. Therefore

$$\sum_\lambda \left(\sum_\rho \zeta_{\lambda\rho}\mu_\rho(x)\right)\left(\sum_\nu \eta_{\lambda\nu} v_\nu(y)\right) = \sum_{\rho\nu}\left(\sum_\lambda \zeta_{\lambda\rho}\eta_{\lambda\nu}\right) u_\rho(x) v_\nu(y) \tag{1.2}$$

which implies $\{u_\lambda | \lambda \vdash r\}$ and $\{v_\lambda | \lambda \vdash r\}$ are dual iff $\prod_{i,j\geq 1} \frac{1}{1-x_i y_j} = \sum_\lambda u_\lambda(x) v\lambda(y)$. □

Corollary 1.5. *Cauchy identity.*
$$\prod_{i,j\geq 1}\frac{1}{1 - x_i y_j} = \sum_\lambda s_\lambda(x) s_\lambda(y).$$

Remark 1.6. This is related to RSK, which gives us a bijection between non-negative integer matrices of finite support with row(A)=α and col(B)=β and $\bigcup_\lambda SSYT(\lambda,\alpha) \times SSYT(\lambda, \beta)$

Remark 1.7. Λ is a self dual Hopf algebra, $\langle \Delta f, g \otimes h\rangle = \langle f, gh\rangle$.

2. k-Schur Functions in the Nilcoxeter algebra

Recall that x commutes with nilcoxeter generators. Let α be a weak composition.

Proposition 2.1. $\sum_{\alpha:\alpha_i \leq k} \not{h}_\alpha x^\alpha = \sum_{\lambda:\lambda_1\leq k} \not{s}_\lambda^{(k)} F_\lambda$

Proof. $s_\lambda^{(k)}$ and F_λ are dual bases, so we have $\sum_{\lambda:\lambda_1\leq k} s_\lambda^{(k)}(y) F_\lambda(x) = \sum_\alpha h_\alpha(y) x^\alpha$ inside $\Lambda_{(k)} \times \Lambda^{(k)}$. Then just lift to the noncommutative setting. □

$F_w = \sum_\alpha \langle A_w, \not{h}_\alpha\rangle x^\alpha$ and by the previous proposition this equals $\sum_\lambda \langle A_w, \not{s}_\lambda^{(k)}\rangle F_\lambda(x)$. Let $a_{w\lambda} = \langle A_w, \not{s}_\lambda^{(k)}\rangle$.

Corollary 2.2. *The coefficient of A_w in $\not{s}_\lambda^{(k)}$ is equal to the coefficient of F_λ in F_w.*

The following theorem was proved by Lam using geometry.

Theorem 2.3. $a_{w\lambda} \in \mathbb{Z}_{\geq 0}$.

Definition 2.4. $\{s_\lambda^{(k)}\}$ form a basis of $\Lambda_{(k)}$, define $s_\lambda^{(k)} s_\mu^{(k)} = \sum_{\nu:\nu_1 \leq k} c_{\lambda\mu}^{\nu,k} s_\nu^{(k)}$. The $c_{\lambda\mu}^{\nu,k}$ are called k-Littlewood-Richardson coefficients.

3. Skew Affine Stanley Symmetric Functions

$$F_{w/v} = \sum_\alpha \langle A_w, \not{h}_\alpha A_v \rangle x^\alpha = F_{wv^{-1}}, \text{ where } w = uv \text{ and } \ell(w) = \ell(u) + \ell(v).$$

Proposition 3.1. $\Delta F_w = F_w(x,y) = \sum_{uv=w:\ell(w)=\ell(u)+\ell(v)} F_u(x) F_v(y)$

Proof. Recall that $F_w(x) = \sum_{\mu:\mu_1 \leq k} k_{w\mu}^{(k)} m_\mu$. Then $\Delta m_\mu = \sum_{\alpha \cup \beta = \mu} m_\alpha \otimes m_\beta$ implies $\Delta F_w = \sum_{\mu:\mu_1 \leq k} k_{w\mu}^{(k)} \sum_{\alpha \cup \beta = \mu} m_\alpha \otimes m_\beta = \sum_{v,\alpha,\beta} k_{w/v,\alpha}^{(k)} k_{v,\beta}^{(k)} m_\alpha \otimes m_\beta = \sum_v F_{w/v} \otimes F_v = \sum_{uv=w} f_u \otimes F_v$. ∎