LECTURE 7: THE CAUCHY IDENTITY

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1. Noncommutative k-Schur Functions

Definition 1.1. (Noncommutative k-Schur Function)
$\not{s}_\lambda^{(k)} = \sum_{\mu:\mu_i \leq k} \gamma_{\lambda\mu} \not{h}_\mu$ where $\not{h}_r = \sum_{J \subsetneq I, |J|=r} A_j^{dec}$ and $\not{h}_\mu = \not{h}_{\mu_1} \not{h}_{\mu_2} \cdots$
Note that the notation for the k-Schur function is $s_\lambda^{(k)} = \sum_{\mu:\mu_i \leq k} \gamma_{\lambda\mu} h_\mu$.

Definition 1.2. (Affine Stanley Symmetric Function)
$\mathcal{F}_w = \sum_\alpha < A_w, \not{h}_{\alpha_1} \not{h}_{\alpha_2} \cdots > x^\alpha$ for all $w \in \tilde{S}_n$ where $< A_w, A_v >= \delta_{wv}$. $\mathcal{F}_\lambda = \sigma_\lambda^{(k)}$, the dual k-Schur function, if w is an affine Grassmanian $w \leftrightarrow \lambda$ is k bounded partition.

2. Cauchy Identity

Let Λ be the ring of symmetric functions.
$h_r = \sum_{\lambda \vdash r} m_\lambda$
$h_0 = m_\emptyset = 1$
$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$ where $\lambda = (\lambda_1, \lambda_2, \cdots) \in P$

Proposition 2.1. *$\lambda \vdash r$ $\alpha = (\alpha_1, \alpha_2, \cdots)$ is a weak composition of r. Then the coefficient $N_{\lambda\alpha}$ of x^α in $h_\lambda = \sum_{\mu \vdash r} N_{\lambda\mu} m_\mu$ is the number of matrices $A = (A_{ij}); i,j \geq 1$ with $(A_{ij}) \in \mathbb{N} \cup \{0\}$ such that $row(A) = \lambda$ and $col(A) = \alpha$.*

Proof. Term x^α in $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$ is obtained by choosing $x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} \cdots$ from each h_{λ_i} such that $\prod_i x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} \cdots = x^\alpha$. This is equivalent to choosing a matrix $A = (A_{ij})$ with $A_{ij} \in \mathbb{N} \cup \{0\}$ with $row(A) = \lambda$ and $col(A) = \alpha$. $\square$

Proposition 2.2. $\prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} = \sum_{\lambda,\mu \in P} N_{\lambda\mu} m_\lambda(x) m_\mu(y) = \sum_{\lambda \in P} m_\lambda(x) h_\lambda(y)$.

Proof. The right equality is clear from the previous proposition. So it is enough to prove the left equality. For each term of the product, we Taylor expand $\frac{1}{1-x_i y_j}$ into a geometric series to obtain a product of Taylor expansions. The monomial $x^\alpha y^\beta$ appearing in $\prod_{i,j \geq 1} \frac{1}{1-x_i y_j}$ corresponds to a matrix $A = (A_{ij}); i,j \geq 1$ with $(A_{ij}) \in \mathbb{N} \cup \{0\}$ such that $\prod_{i,j \geq 1} (x_i y_j)^{A_{ij}} = x^{row(A)} y^{col(A)} = x^\alpha y^\beta$. $\square$

Definition 2.3. Two bases $\{u_\lambda\}, \{v_\lambda\}$ of Λ are dual if $< u_\lambda, v_\lambda >= \delta_{\lambda\mu}$

Proposition 2.4. *$\{u_\lambda | \lambda \vdash r\}, \{v_\lambda | \lambda \vdash r\}$ are bases of Λ^r (graded piece of λ of degree r).*
$\{u_\lambda\}, \{v_\lambda\}$ are dual bases $\Leftrightarrow \sum_{\lambda \in P} u_\lambda(x) v_\lambda(y) = \prod_{i,j \geq 1} \frac{1}{1-x_i y_j} = \sum_{\lambda \in P} m_\lambda(x) h_\lambda(y)$.

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Proof. Note that the last equality has been proven in the previous proposition.
Write $m_\lambda = \sum_\rho \zeta_{\lambda\rho} u_\rho$ and $h_\mu = \sum_\nu \eta_{\mu\nu} v_\nu$.
Then $\delta_{\lambda\mu} =< m_\lambda, h_\mu >= \sum_{\rho,\nu} \zeta_{\lambda\rho}\eta_{\mu\nu} < u_\rho, v_\nu >$
Let $A_{\rho\nu} =< u_\rho, v_\nu >$. Fixed r, ρ and η are matrices indexed by P_r. $\Leftrightarrow I = \zeta A \eta^t$.
Hence $\{u_\lambda\}, \{v_\lambda\}$ are dual. $\Leftrightarrow A = I \Leftrightarrow I = \zeta\eta^t \Leftrightarrow I = \zeta^t\eta \Leftrightarrow \delta_{\rho\nu} = \sum_\lambda \zeta_{\lambda\rho}\eta_{\lambda\nu}$
Now $\prod_{i,j\geq 1} \frac{1}{1-x_i y_j} = \sum_{\lambda\in P} m_\lambda(x) h_\lambda(y) = \sum_\lambda (\sum_\rho \zeta_{\lambda\rho} u_\rho(x))(\sum_\nu \eta_{\lambda\nu} v_\nu(y))$
$= \sum_{\rho,\nu}(\sum_\lambda \zeta_{\lambda\rho}\eta_{\lambda\nu}) u_\rho(x) v_\nu(y) \Rightarrow \{u_\lambda\}, \{v_\lambda\}$ are dual. $\Leftrightarrow \prod_{i,j\geq 1} \frac{1}{1-x_i y_j} = \sum_{\lambda\in P} u_\lambda(x) v_\lambda(y)$. □

Corollary 2.5. *(Cauchy Identity)* $\prod_{i,j\geq 1} \frac{1}{1-x_i y_j} = \sum_\lambda s_\lambda(x) s_\lambda(y)$

Remark 2.6. By Robinson-Schensted-Knuth (RSK) bijection:
$\varphi : \mathcal{A} \longleftrightarrow \cup_\lambda SSYT(\lambda, \alpha) \times SSYT(\lambda, \beta)$
where $\mathcal{A} = \{$ matrices A of nonnegative integer entries and finite support $\}$
$row(A) = \alpha$ and $col(A) = \beta$
SSYT = Semi Standard Young Tableau.

Remark 2.7. Λ is a self-dual Hopf-algebra under the Hall inner product since $< \Delta f, g \otimes h >=< f, gh >$ for $f, g, h \in \Lambda$ where $\Delta : \Lambda \to \Lambda \otimes \Lambda$ is the coproduct.

3. k-Schur Functions in nil-Coxeter algebra

Proposition 3.1. $\sum_{\alpha:\alpha_i\leq k} \not{h}_\alpha x^\alpha = \sum_{\lambda:\lambda_i\leq k} \not{s}_\lambda^{(k)} \mathcal{F}_\lambda(x)$

Proof. $s_\lambda^{(k)}$ and $\mathcal{F}_\lambda$ are dual bases.
By the Cauchy Identity, $\sum_\lambda s_\lambda^{(k)}(y)\mathcal{F}_\lambda(x) = \sum_\alpha h_\alpha(y) x^\alpha$. □

For affine Stanley symmetric functions:
$\mathcal{F}_w = \sum_\alpha < A_w, \not{h}_\alpha > x^\alpha = \sum_\lambda < A_w, \not{s}_\lambda^{(k)} > \mathcal{F}_\lambda(x)$ by previous proposition.
The coefficient of A_w in $\not{s}_\lambda^{(k)}$ equals the coefficient of $\mathcal{F}_\lambda$ in $\mathcal{F}_w$, $w \in \tilde{S}_n$.

Theorem 3.2. *(Lam)* $a_{w\lambda} :=< A_w, \not{s}_\lambda^{(k)} >\in \mathbb{N} \cup \{0\}$

Definition 3.3. $\{s_\lambda^{(k)}\}$ forms a basis of $\Lambda_{(k)}$. Define $s_\lambda^{(k)} s_\mu^{(k)} = \sum_{\nu:\nu_i\leq k} c_{\lambda\mu}^{\nu,k} s_\nu^{(k)}$ where $c_{\lambda\mu}^{\nu,k}$ is the k-Littlewood Richardson coefficient.

For skew affine Stanley symmetric functions:
$\mathcal{F}_{w/v} = \sum_\alpha < A_w, \not{h}_\alpha A_v > x^\alpha = \mathcal{F}_{wv^{-1}}$
$w = uv$, $l(w) = l(u) + l(v)$

Proposition 3.4. *(Coproduct)* $\Delta\mathcal{F}_w = \mathcal{F}_w(X, Y) = \sum_{uv=w} \mathcal{F}_u(X)\mathcal{F}_v(Y)$

Proof. $\mathcal{F}_w(X) = \sum_{\mu:\mu_i\leq k} K_{w\mu}^{(k)} m_\mu$ and $\Delta m_\mu = \sum_{\alpha\cup\beta=\mu} m_\alpha \otimes m_\beta$ where α, β are partitions.
Then $\Delta\mathcal{F}_w = \sum_{\mu:\mu_i\leq k} K_{w\mu}^{(k)} \sum_{\alpha\cup\beta=\mu} m_\alpha \otimes m_\beta = \sum_{v,\alpha,\beta} K_{w/v,\alpha}^{(k)} K_{v,\beta}^{(k)} m_\alpha \otimes m_\beta$
$= \sum_v \mathcal{F}_{w/v} \otimes \mathcal{F}_v = \sum_{uv=w} \mathcal{F}_u \otimes \mathcal{F}_v$ □