LECTURE 7: THE CAUCHY IDENTITY

JASON OCHIAI

1. NONCOMMUTATIVE K-SCHUR FUNCTIONS

Definition 1.1. (Noncommutative k-Schur Function)

\[ s^{(k)}_\lambda = \sum_{\mu, \nu \leq \lambda} \gamma_{\nu \lambda} h_\mu \]

where \( h_r = \sum_{j \leq i, j = r} A_{ij}^{dec} \) and \( h_\mu = \hat{h}_1 h_2 \cdots \)

Note that the notation for the k-Schur function is \( s^{(k)}_\lambda = \sum_{\mu, \nu \leq \lambda} \gamma_{\nu \lambda} h_\mu \).

Definition 1.2. (Affine Stanley Symmetric Function)

\[ F_w = \prod_{\nu \leq \lambda} < A_w, h_\nu h_\mu \cdots > x^\alpha \]

for all \( w \in S_n \) where \( < A_w, A_v > = \delta_{wv} \).

\[ F_\lambda = s^{(k)}_\lambda \), the dual k-Schur function, if \( w \) is an affine Grassmanian \( w \leftrightarrow \lambda \) is a \( k \) bounded partition.

2. CAUCHY IDENTITY

Let \( \Lambda \) be the ring of symmetric functions.

\[ h_r = \sum_{\lambda \vdash r} m_\lambda \]

\[ h_0 = m_0 = 1 \]

\[ h_\lambda = h_\lambda h_\lambda \cdots \] where \( \lambda = (\lambda_1, \lambda_2, \cdots) \in P \)

Proposition 2.1. \( \lambda \vdash r \) \( \alpha = (\alpha_1, \alpha_2, \cdots) \) is a weak composition of \( r \). Then the coefficient \( N_{\lambda \mu} \) of \( x^\alpha \) in \( h_\lambda = \sum_{\rho \vdash r} N_{\lambda \rho} m_\mu \) is the number of matrices \( A = (A_{ij}); i, j \geq 1 \) with \( (A_{ij}) \in N \cup \{0\} \) such that \( row(A) = \lambda \) and \( col(A) = \alpha \).

Proof. Term \( x^\alpha \) in \( h_\lambda = h_\lambda h_\lambda \cdots \) is obtained by choosing \( x_{i_1}^{\alpha_{i_1}} x_{i_2}^{\alpha_{i_2}} \cdots \) from each \( h_\lambda \), such that \( \prod_i x_i^{\alpha_i} x_{i_2}^{\alpha_{i_2}} \cdots = x^\alpha \). This is equivalent to choosing a matrix \( A = (A_{ij}) \) with \( A_{ij} \in N \cup \{0\} \) with \( row(A) = \lambda \) and \( col(A) = \alpha \).

\[ \prod_{i,j \geq 1} \frac{1}{1-x_i y_j} = \sum_{\lambda, \mu \in P} N_{\lambda \mu} m_\lambda(x) m_\mu(y) = \sum_{\lambda \in P} m_\lambda(x) h_\lambda(y). \]

Proof. The right equality is clear from the previous proposition. So it is enough to prove the left equality. For each term of the product, we Taylor expand \( \frac{1}{1-x_i y_j} \) into a geometric series to obtain a product of Taylor expansions. The monomial \( x^\alpha y^\beta \) appearing in \( \prod_{i,j \geq 1} \frac{1}{1-x_i y_j} \) corresponds to a matrix \( A = (A_{ij}); i, j \geq 1 \) with \( (A_{ij}) \in N \cup \{0\} \) such that \( \prod_{i,j \geq 1} (x_i y_j)^{A_{ij}} = x^{row(A)} y^{col(A)} = x^\alpha y^\beta \).

Definition 2.3. Two bases \( \{u_\lambda\}, \{v_\lambda\} \) of \( \Lambda \) are dual if \( < u_\lambda, v_\lambda > = \delta_{\lambda \mu} \)

Proposition 2.4. \( \{u_\lambda \mid \lambda \vdash r\}, \{v_\lambda \mid \lambda \vdash r\} \) are bases of \( \Lambda^r \) (graded piece of \( \lambda \) of degree \( r \)).

\[ \{u_\lambda\}, \{v_\lambda\} \] are dual bases \( \Leftrightarrow \sum_{\lambda \in P} u_\lambda(x) v_\lambda(y) = \prod_{i,j \geq 1} \frac{1}{1-x_i y_j} = \sum_{\lambda \in P} m_\lambda(x) h_\lambda(y) \).

Date: October 22, 2012.
Proof. Note that the last equality has been proven in the previous proposition.

Write \( m_\lambda = \sum \zeta_\rho u_\rho \) and \( h_\mu = \sum \eta_\nu v_\nu \).

Then \( \delta_\mu = \langle m_\lambda, h_\mu \rangle = \sum \zeta_\rho \eta_\nu \langle u_\rho, v_\nu \rangle \).

Let \( A_{\rho\nu} = \langle u_\rho, v_\nu \rangle \). Fixed \( r, p \) and \( q \) are matrices indexed by \( P_r, \Rightarrow I = \zeta A_\eta \).

Hence \( \{ u_\lambda \}, \{ v_\lambda \} \) are dual. \( \Rightarrow A = I \Leftrightarrow \lambda = \zeta \eta \Leftrightarrow \delta_\mu = \sum \zeta_\rho \eta_\nu \langle u_\rho, v_\nu \rangle \).

Now \( \prod_{i,j \geq 1} \frac{1}{1-x_{ij}} = \sum_{\lambda \in \mathcal{P}} m_\lambda(x)h_\lambda(y) = \sum_{\lambda}(\sum \zeta_\rho \eta_\nu u_\rho(x))v_\nu(y) \).

Then \( \sum \zeta_\rho \eta_\nu u_\rho(x)v_\nu(y) \Rightarrow \{ u_\lambda \}, \{ v_\lambda \} \) are dual. \( \Rightarrow \prod_{i,j \geq 1} \frac{1}{1-x_{ij}} = \sum_{\lambda \in \mathcal{P}} u_\lambda(x)v_\lambda(y). \) \( \square \)

Corollary 2.6. (Cauchy Identity) \( \prod_{i,j \geq 1} \frac{1}{1-x_{ij}} = \sum_\lambda s_\lambda(x)s_\lambda(y) \).

Remark 2.6. By Robinson-Schensted-Knuth (RSK) bijection:
\[
\text{(Cauchy Identity)}
\]

Theorem 2.6. (Lam) \( a_{\mu \lambda} \vdash < A_{\mu \lambda} > \in \mathbb{N} \cup \{ 0 \} \)

Definition 3.3. \( \{ s_\lambda^{(k)} \} \) forms a basis of \( \Lambda_\mu \). Define \( s_\lambda^{(k)} s_\mu^{(k)} = \sum_{\nu \leq k} c_{\nu \mu}^{(k)} s_\nu^{(k)} \)

where \( c_{\nu \mu}^{(k)} \) is the k-Littlewood Richardson coefficient.

For skew affine Stanley symmetric functions:
\[
F_{w/v} = \sum_{\alpha} < A_{w,v}, h_\alpha \rangle > x^\alpha = F_{w/v} \quad w = wv, l(w) = l(u) + l(v)
\]

Proposition 3.4. (Coproduct) \( \Delta F_w = F_{w}(X, Y) = \sum_{w=v} F_u(X)F_v(Y) \)

Proof. \( F_w(X) = \sum_{\mu \leq k} K^{(k)}_{\mu \lambda} m_\mu \) and \( \Delta m_\mu = \sum_{\alpha \beta = \mu} m_\alpha \otimes m_\beta \) where \( \alpha, \beta \) are partitions.

Then \( \Delta F_v = \sum_{\mu \leq k} K^{(k)}_{\mu \lambda} \sum_{\alpha \beta = \mu} m_\alpha \otimes m_\beta = \sum_{v \alpha, \beta} K^{(k)}_{w/v, \alpha} K^{(k)}_{v, \beta} m_\alpha \otimes m_\beta \) \( \square \)