**LECTURE 9: STRONG MARKED TABLEAUX**

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**Proposition 0.1** (LMMS). Let \( \tau, \kappa \) be \((k+1)\)-cores, \( \Rightarrow_k \kappa \), marking of \( \kappa/\tau \) at diagonal \( j-1 \), \( i \) diagonal of the tail of the marked ribbon. Let \( w = w_\tau \) and \( u = u_\kappa \), then the following hold:

1. \( w^{-1}(i) \leq 0 < w^{-1}(j) \)
2. \( t_{ij}w = u \) (note that \( t_{ij} \) makes sense because the ribbon is the right length, otherwise the ribbon could be removed)
3. the number of connected ribbons below the marked one is \((-w^{-1}(i) - a) / n\) where \( a = w^{-1}(i) \mod n \)
4. the number of connected ribbons above the marked one is \((w^{-1}(j) - b) / n\) where \( b = w^{-1}(j) \mod n \)

**Example 0.2.** \( n = 3 \), \( \tau = (5,3,1) \), \( w = s_1 s_0 s_2 s_1 s_0 \), and \( w^{-1} = [4,-3,5] \) (as an exercise, start with the empty partition and apply \( s_i \) actions according to \( w \) to get \( \tau \)). Then \( t_{-1,0} = t_{2,3} = t_{5,6} = s_2 \), \( \kappa = t_{-1,0} \tau = (6,4,2) \), and note that \( \tau \Rightarrow_k \kappa \).

There are three ways of marking, by picking any of the three new boxes labeled with \( x \). The highest has diagonal \(-1 \leftrightarrow t_{-1,0} \), the next has diagonal \( 2 \leftrightarrow t_{2,3} \) and the last has diagonal \( 5 \leftrightarrow t_{5,6} \). Note that these are different ways of writing \( t_{-1,0} \). If \( j = 0 \), \( i = -1 \), \( t_{-1,0} \) then the number of connected components equals \( 1 + \) number below + number above \( = 1 + (-w^{-1}(i) + w^{-1}(j) - a - b) / n \).

**Definition 0.3.** \( \kappa, \tau \) \((k+1)\)-cores, \( \tau \subseteq \kappa \). \( \kappa/\tau \) is a strong marked horizontal strip if there exists a sequence of partitions \( \Rightarrow_k \tau^{(1)} \Rightarrow_k \tau^{(2)} \Rightarrow_k \cdots \Rightarrow_k \tau^{(r)} = \kappa \) with markings \( c_1, \ldots, c_r \) where \( c_i \) is the diagonal of the head of the marked ribbon in \( \tau^{(i)}/\tau^{(i-1)} \) and \( c_1 < c_2 < \cdots < c_r \).

**Example 0.4.** \( k = 3 \),

is not a strong marked horizontal strip when all the boxes marked \( x \) are picked because \( c_1 = -1 \) and \( c_2 = -2 \). If you pick the box marked * it works because \( c_1 = 1, c_2 = 2, c_3 = 3 \).

**Remark 0.5.** \( w, w' \in S_n \), \( \ell(w') = \ell(w) + 1 \). \( w' \) covers \( w \) in weak (left) order iff there exists \( s_i \) s.t. \( s_i w = w' \). \( w' \) covers \( w \) in strong (or Bruhat) order iff there exists \( t_{ij} \) s.t. \( t_{ij} w = w' \).

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1. Strong Marked Tableaux and the Monomial Expansion of k-Schur Functions

Recall $F_\lambda = \sigma_\lambda^{(k)} = \sum_{\mu; \mu \leq k} k_{\lambda, \mu}^{(k)} m_\mu$ where $k_{\lambda, \mu}^{(k)}$ is the $k$-Kostka matrix, which is equal to the number of weak $k$-tableaux of shape $\lambda$ and content $\mu$.

**Example 1.1.** Using $k$-bounded, $h_1\sigma_{321}^{(3)} = 2\sigma_{31}^{(3)} + \sigma_{322}^{(3)} + \sigma_{3111}^{(3)}$. Note that the multiplicities can be greater than 1 and $(3, 2, 1) \not\subseteq (3, 1, 1, 1)$. However, if we use 4-cores we have

\[
\begin{align*}
321 & \rightarrow \begin{array}{c}
\circ,\circ,\circ \\
\circ,\circ,\circ \\
\circ,\circ,\circ \\
\end{array} \\
331 & \rightarrow \begin{array}{c}
\bullet,\bullet,\circ \\
\circ,\bullet,\circ \\
\circ,\circ,\circ \\
\end{array} \\
322 & \rightarrow \begin{array}{c}
\circ,\circ,\bullet \\
\bullet,\circ,\circ \\
\circ,\circ,\circ \\
\end{array} \\
3211 & \rightarrow \begin{array}{c}
\bullet,\circ,\circ,\circ \\
\circ,\bullet,\circ,\circ \\
\circ,\circ,\bullet,\circ \\
\circ,\circ,\circ,\bullet \\
\end{array} \\
3111 & \rightarrow \begin{array}{c}
\bullet,\bullet,\circ,\circ \\
\circ,\bullet,\bullet,\circ \\
\circ,\circ,\bullet,\bullet \\
\end{array}
\end{align*}
\]

so we have containment as shown by the shading and we see the coefficients as the number of ribbons.

**Theorem 1.2** (LLMS). $\lambda$-k-bounded,

\[
(1.1) \quad h_r\sigma_\lambda^{(k)} = \sum_{(\kappa^{(r)}, c_r)} \sigma^{(k)}_{p(k(r))}
\]

where the sum is over all strong marked horizontal strips $\kappa^{(*)} = (c(\lambda) = \kappa^{(0)} \Rightarrow j \cdots \Rightarrow \kappa^{(r)})$ with markings $c_r = (c_1 < c_2 < \ldots < c_r)$.

**Definition 1.3.** A strong marked tableaux of shape $\lambda \vdash m$ (k-bounded) and content $\alpha = (\alpha_1, \ldots, \alpha_d)$ $\alpha_1 + \cdots + \alpha_d = m$ is a sequence $\kappa^{(0)} \Rightarrow \kappa^{(1)} \Rightarrow \cdots \Rightarrow \kappa^{(m)} = c(\lambda)$ and markings $c_r = (c_1, \ldots, c_m)$ s.t. $(\kappa^{(0)}, \ldots, \kappa^{(v+\alpha_r)})$ with markings $(c_v, \ldots, c_v + \alpha_r) \nu = \alpha_1 + \cdots + \alpha_r - 1$ is a strong marked horizontal strip $\forall \nu \leq r \leq d$.

**Remark 1.4.** Strong marked covers correspond to left multiplicatin by $t_{ij}$.

\[
t_{a+b} \cdots t_{a+1+j_a} t_i t_{a+j_a}
\]

is a strong marked horizontal strip if $j_a < j_{a+1} < \cdots < j_{a+b}$.

**Definition 1.5.** $K_{\lambda, \mu}^{(k)}$ equals the number of strong marked tableaux of shape $\lambda$ and weight $\mu$.

In the weak case $h_\mu = \cdots h_{\mu_2} h_{\mu_1} s_\emptyset$ and you do the Pieri rule on each piece. Now $h_\mu = \cdots h_{\mu_2} h_{\mu_1} s_\emptyset = \sum_{\lambda: \lambda \leq k} K_{\lambda, \mu}^{(k)} c_{\lambda, \mu}^{(k)}$ and $s_\emptyset^{(k)} = \sum_{\mu: \mu \leq k} K_{\lambda, \mu}^{(k)} m_\mu$.

2. Littlewood-Richardson Rule

\[
s_\lambda s_\mu = \sum_\nu c_{\lambda, \mu}^{\nu} s_\nu \quad c_{\lambda, \mu}^{\nu} \text{ is the Littlewood-Richardson coefficient and equals the number of skew tableaux of shape } \nu/\lambda \text{ and weight } \mu \text{ s.t. the row reading word is a reverse lattice word.}
\]
Example 2.1. $\lambda = 21$, $\mu = 321$, $\nu = 432$ then $\nu/\lambda = \begin{array}{c} 2 \\ 3 \\ 1 \\ 2 \\ 1 \end{array}$ is valid.

A row reading word goes from top to bottom and left to right: 23 12 11. A reverse lattice word has weakly more 1 entries than 2 entries, weakly more 2 entries than 3 entries . . . at each step reading from right to left (the weight needs to be a partition).

has reading word 12 13 12 which is not reverse lattice because the first number is 2.

Remark 2.2. Reverse lattice words correspond to highest weight crystal elements.

3. Crystals

Crystal elements are tableaux (or words) over an alphabet $\{1, 2, \ldots, n\}$ (in this section $n$ has no relation to $k$). For these crystal elements we have Kashiwara operators $f_i, e_i, s_i$ for $1 \leq i < n$. In terms of words they act as follows. First successively bracket $i + 1$ and $i$ ($i + 1 \to [i, i \to]$) and ignore all paired $i, i + 1$ as well as all $j \neq i, i + 1$. What will remain is $i^a(i + 1)^b$. Then

\begin{align*}
e_i(i^a(i + 1)^b) & = \begin{cases} i^{a+1}(i + 1)^{b-1} & b > 0 \\ 0 & b = 0 \end{cases} \\
f_i(i^a(i + 1)^b) & = \begin{cases} i^{a-1}(i + 1)^{b+1} & a > 0 \\ 0 & a = 0 \end{cases} \\
s_i(i^a(i + 1)^b) & = i^b(i + 1)^a \end{align*}

Example 3.1. For the alphabet $\{1, 2, 3\}$, $211 \leftrightarrow \begin{array}{c} 2 \\ 1 \\ 1 \end{array}$. The crystal graph has $w \Rightarrow w'$ if $w' = f_i w$. $e_i(211) = 0$ for all $i$ so 211 is called highest weight.