LECTURE 9: STRONG MARKED TABLEAUX

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Proposition 0.1 (LMMS). *Let τ, κ be (k+1)-cores, $\tau \Rightarrow_k \kappa$, marking of κ/τ at diagonal $j-1$, i diagonal of the tail of the marked ribbon. Let $w = w_\tau$ and $u = u_\kappa$, then the following hold:*

(1) $w^{-1}(i) \leq 0 < w^{-1}(j)$

(2) $t_{ij}w = u$ (note that t_{ij} makes sense because the ribbon is the right length, otherwise the ribbon could be removed)

(3) the number of connected ribbons below the marked one is $(-w^{-1}(i) - a)/n$ where $a = w^{-1}(i)$ mod n

(4) the number of connected ribbons above the marked one is $(w^{-1}(j) - b)/n$ where $b = w^{-1}(j)$ mod n

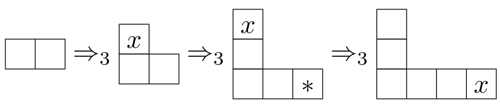
Example 0.2. $n = 3$, $\tau = (5,3,1)$, $w = s_1s_0s_2s_1s_0$, and $w^{-1} = [4,-3,5]$ (as an exercise, start with the empty partition and apply s_i actions according to w to get τ). Then $t_{-1,0} = t_{2,3} = t_{5,6} = s_2$. $\kappa = t_{-1,0}\tau = (6,4,2)$, and note that $\tau \Rightarrow_k \kappa$.

1	x				
2	0	1	x		
0	1	2	0	1	x

There are three ways of marking, by picking any of the three new boxes labeled with x. The highest has diagonal $-1 \leftrightarrow t_{-1,0}$, the next has diagonal $2 \leftrightarrow t_{2,3}$ and the last has diagonal $5 \leftrightarrow t_{5,6}$. Note that these are different ways of writing $t_{-1,0}$. If $j = 0, i = -1, t_{-1,0}$ then the number of connected components equals $1 +$ number below $+$ number above $= 1 + (-w^{-1}(i) + w^{-1}(j) - a - b)/n = 1 + (6+2-0-2)/3 = 3$.

Definition 0.3. κ, τ (k+1)-cores, $\tau \subseteq \kappa$. κ/τ is a strong marked horizontal strip if there exists a sequence of partitions $\tau \Rightarrow_k \tau^{(1)} \Rightarrow_k \tau^{(2)} \Rightarrow_k \ldots \Rightarrow_k \tau^{(r)} = \kappa$ with markings $c_1, \ldots, c_r$ where c_i is the diagonal of the head of the marked ribbon in $\tau^{(i)}/\tau^{(i-1)}$ and $c_1 < c_2 < \cdots < c_r$.

Example 0.4. $k = 3$,



is not a strong marked horizontal strip when all the boxes marked x are picked because $c_1 = -1$ and $c_2 = -2$. If you pick the box marked $*$ it works because $c_1 = 1, c_2 = 2, c_3 = 3$.

Remark 0.5. $w, w' \in S_n$ $\ell(w') = \ell(w) + 1$. w' covers w in weak (left) order iff there exists s_i s.t. $s_iw = w'$. w' covers w in strong (or Bruhat) order iff there exists t_{ij} s.t. $t_{ij}w = w'$.

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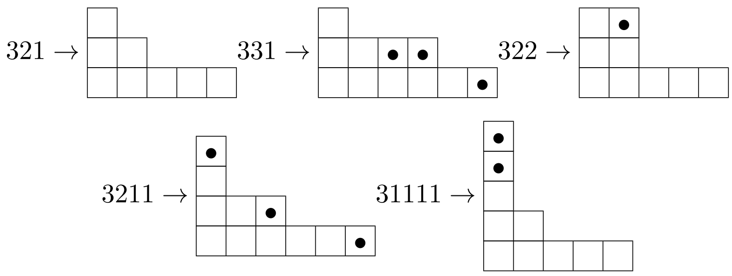
1. Strong Marked Tableaux and the Monomial Expansion of k-Schur Functions

Recall $F_\lambda = \sigma_\lambda^{(k)} = \sum_{\mu:\mu_1 \leq k} k_{\lambda,\mu}^{(k)} m_\mu$ where $k_{\lambda,\mu}^{(k)}$ is the k-Kostka matrix, which is equal to the number of weak k-tableaux of shape λ and content μ.

Example 1.1. Using k-bounded, $h_1\sigma_{321}^{(3)} = 2\sigma_{331}^{(3)} + \sigma_{322}^{(3)} + \sigma_{3211}^{(3)} + \sigma_{31111}^{(3)}$. Note that the multiplicities can be greater than 1 and $(3,2,1) \not\subseteq (3,1,1,1,1)$. However, if we use 4-cores we have



so we have containment as shown by the shading and we see the coefficients as the number of ribbons.

Theorem 1.2 (LLMS). *λ k-bounded,*

$$h_r\sigma_\lambda^{(k)} = \sum_{(\kappa^{(*)},c_*)} \sigma_{p(k^{(r)})}^{(k)} \tag{1.1}$$

where the sum is over all strong marked horizontal strips $\kappa^{(*)} = (c(\lambda) = \kappa^{(0)} \Rightarrow_k \ldots \Rightarrow_k \kappa^{(r)})$ with markings $c_* = (c_1 < c_2 < \ldots < c_r)$.

Definition 1.3. A strong marked tableaux of shape $\lambda \vdash m$ (k-bounded) and content $\alpha = (\alpha_1, \ldots, \alpha_d)$ $\alpha_1 + \ldots + \alpha_d = m$ is a sequence $\kappa^{(0)} \Rightarrow_k \kappa^{(1)} \Rightarrow_k \ldots \Rightarrow_k \kappa^{(m)} = c(\lambda)$ and markings $c_* = (c_1, \ldots, c_m)$ s.t. $(\kappa^{(v)}, \ldots, \kappa^{(v+\alpha_r)})$ with markings $(c_{v+1}, \ldots, c_{v+\alpha_r})$ $v = \alpha_1 + \ldots + \alpha_{r-1}$ is a strong marked horizontal strip $\forall 1 \leq r \leq d$.

Remark 1.4. Strong marked covers correspond to left multiplicatin by t_{ij}.

$$t_{i_{a+b}j_{a+b}} \cdots t_{i_{a+1}j_{a+1}} t_{i_a j_a}$$

is a strong marked horizontal strip if $j_a < j_{a+1} < \cdots < j_{a+b}$.

Definition 1.5. $\mathbf{K}_{\lambda\mu}^{(k)}$ equals the number of strong marked tableaux of shape λ and weight μ.

In the weak case $h_\mu = \cdots h_{\mu_2} h_{\mu_1} s_\emptyset$ and you do the Pieri rule on each piece. Now $h_\mu = \cdots h_{\mu_2} h_{\mu_1} \sigma_\emptyset^{(k)} = \sum_{\lambda:\lambda_1 \leq k} \mathbf{K}_{\lambda\mu}^{(k)} \sigma_\lambda^{(k)}$, $\langle s_\lambda^{(k)}, h_\mu \rangle = \mathbf{K}_{\lambda\mu}^{(k)}$, and $s_\lambda^{(k)} = \sum_{\mu:\mu_1 \leq k} \mathbf{K}_{\lambda\mu}^{(k)} m_\mu$.

2. Littlewood-Richardson Rule

$s_\lambda s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu$. $c_{\lambda\mu}^\nu$ is the Littlewood-Richardson coefficient and equals the number of skew tableaux of shape ν/λ and weight μ s.t. the row reading word is a reverse lattice word.

Example 2.1. $\lambda = 21$, $\mu = 321$, $\nu = 432$ then $\nu/\lambda = \begin{array}{|c|c|cc} \hline 2 & 3 & & \\ \hline & 1 & 2 & \\ \hline & & 1 & 1 \\ \hline \end{array}$ is valid.

A row reading word goes from top to bottom and left to right: 23 12 11. A reverse lattice word has weakly more 1 entries than 2 entries, weakly more 2 entries than 3 entries ... at each step reading from right to left (the weight needs to be a partition).

$\begin{array}{cccc} \hline 1 & 2 & & \\ \hline & 1 & 3 & \\ \hline & & 1 & 2 \\ \hline \end{array}$ has reading word 12 13 12 which is not reverse lattice because the first number is 2.

Remark 2.2. Reverse lattice words correspond to highest weight crystal elements.

3. Crystals

Crystal elements are tableaux (or words) over an alphabet $\{1, 2, \dots, n\}$ (in this section n has no relation to k). For these crystal elements we have Kashiwara operators f_i, e_i, s_i for $1 \leq i < n$. In terms of words they act as follows. First successively bracket $i+1$ and i ($i+1 \to [, i \to]$) and ignore all paired $i, i+1$ as well as all $j \neq i, i+1$. What will remain is $i^a(i+1)^b$. Then

$$e_i(i^a(i+1)^b) = \begin{cases} i^{a+1}(i+1)^{b-1} & b > 0 \\ 0 & b = 0 \end{cases} \tag{3.1}$$

$$f_i(i^a(i+1)^b) = \begin{cases} i^{a-1}(i+1)^{b+1} & a > 0 \\ 0 & a = 0 \end{cases} \tag{3.2}$$

$$s_i(i^a(i+1)^b) = i^b(i+1)^a \tag{3.3}$$

Example 3.1. For the alphabet $\{1, 2, 3\}$, $211 \leftrightarrow \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 1 \\ \hline \end{array}$. The crystal graph has $w \xrightarrow{i} w'$ if $w' = f_i w$. $e_i(211) = 0$ for all i so 211 is called highest weight.