MAT 150A University of California Fall 2014

Homework 4

due October 31, 2014 in class

Read: Artin 2.7-2.10

1. Let S be a set of groups. Prove that the relation $G \sim H$ if G is isomorphic to H is an equivalence relation on S.

2. Let H be a subgroup of a group G. Prove that the relation defined by the rule $a \sim b$ if $b^{-1}a \in H$ is an equivalence relation on G.

3. Determine the index $[\mathbb{Z} : n\mathbb{Z}]$.

4. Let H, K be subgroups of a group G of orders $3, 5$ respectively. Prove that $H \cap K = \{1\}$.

5. Artin 2.8.2 (pg. 72) In the additive group $\mathbb{R}^m$ of vectors, let W be the set of solutions of a system of homogeneous linear equations $Ax = 0$. Show that the set of solutions of an inhomogeneous system $Ax = b$ is either empty, or else forms a coset of W.

6. (a) Prove that every subgroup of index 2 is normal.
   (b) Give an example of a subgroup of index 3 which is not normal.

7. Let G and G' be groups. What is the order of the group $G \times G'$?

8. Is the symmetric group S_3 a direct product of nontrivial groups?

9. Recall that the dihedral group D_n is generated by the counterclockwise rotation x and a reflection y:

   $$D_n = \langle x, y \mid x^n = y^2 = 1, xy = yx^{n-1} \rangle.$$

   Use the generators and relations for D_n to show that every element of D_n, which is not a power of x has order 2. Deduce that D_n is generated by the two elements y and yx, both of which have order 2.