Homework 9

due December 4, 2015

1. Rosen 7.3 #11,13, pg. 267

Let n be a positive integer. We say that n is deficient if $\sigma(n) < 2n$ and we say that n is abundant if $\sigma(n) > 2n$.
(a) Show that there are infinitely many deficient numbers.
(b) Show that there are infinitely many odd abundant numbers. (*Hint:* Look at integers of the form $n = 3^k \cdot 5 \cdot 7$).

2. Rosen 8.5 #2, pg. 336

Show that if $a_1, a_2, \ldots, a_n$ is a super-increasing sequence, then $a_j \geq 2^{j-1}$ for $j = 1, 2, \ldots, n$.

3. Rosen 8.5 #3, pg. 336

Show that the sequence $a_1, a_2, \ldots, a_n$ is super-increasing if $a_{j+1} > 2a_j$ for $j = 1, 2, \ldots, n-1$.

4. Rosen 8.5 #10,12, pg. 337

A multiplicative knapsack problem is a problem of the following type: Given positive integers $a_1, a_2, \ldots, a_n$ and a positive integer P, find the subset, or subsets, of these integers with product P. Or equivalently, find all solutions of

$$P = a_1^{x_1} a_2^{x_2} \cdots a_n^{x_n},$$

where $x_j = 0$ or 1 for $j = 1, 2, \ldots, n$.

(a) Find all products of subsets of the integers $2, 3, 5, 6, 10$ equal to 60.

(b) Show that if the integers $a_1, a_2, \ldots, a_n$ are pairwise relatively prime, then the multiplicative knapsack problem $P = a_1^{x_1} a_2^{x_2} \cdots a_n^{x_n}$, $x_j = 0$ or 1 for $j = 1, 2, \ldots, n$ is easily solved from the prime factorizations of the integers $P, a_1, a_2, \ldots, a_n$, and show that if there is a solution, then it is unique.

5. Prove that for $n \in \mathbb{N}$ with $n > 1$

$$(n-1)! \equiv \begin{cases} -1 & \text{if } n \text{ is prime} \\ 2 & \text{if } n = 4 \\ 0 & \text{otherwise} \end{cases} \pmod{n}.$$