Homework 4
due October 23, 2015 in class

Read: Artin 2.7-2.10

1. Let $S$ be a set of groups. Prove that the relation $G \sim H$ if $G$ is isomorphic to $H$ is an equivalence relation on $S$.

2. Let $H$ be a subgroup of a group $G$. Prove that the relation defined by the rule $a \sim b$ if $b^{-1}a \in H$ is an equivalence relation on $G$.

3. Determine the index $[\mathbb{Z} : n\mathbb{Z}]$.

4. Let $H, K$ be subgroups of a group $G$ of orders 3, 5 respectively. Prove that $H \cap K = \{1\}$.

5. Artin 2.8.2 (pg. 72)
   In the additive group $\mathbb{R}^m$ of vectors, let $W$ be the set of solutions of a system of homogeneous linear equations $Ax = 0$. Show that the set of solutions of an inhomogeneous system $Ax = b$ is either empty, or else forms a coset of $W$.

6. (a) Prove that every subgroup of index 2 is normal.
   (b) Give an example of a subgroup of index 3 which is not normal.

7. Let $G$ and $G'$ be groups. What is the order of the group $G \times G'$?

8. Is the symmetric group $S_3$ a direct product of nontrivial groups?

9. Recall that the dihedral group $D_n$ is generated by the counterclockwise rotation $x$ and a reflection $y$:

   $$D_n = \langle x, y \mid x^n = y^2 = 1, xy = yx^{n-1} \rangle.$$

   Use the generators and relations for $D_n$ to show that every element of $D_n$, which is not a power of $x$ has order 2. Deduce that $D_n$ is generated by the two elements $y$ and $yx$, both of which have order 2.