THE GAME SET® AS $\mathbb{F}_3^4$

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Abstract. In this paper we construct an isomorphism between the card game Set® and the four-dimensional vector space over the three element field, $\mathbb{F}_3$, to draw various results about the game. By creating a one-to-one and onto correspondence between the cards and points in $\mathbb{F}_3^4$, we find that a collectable set is in fact a line in the vector space. Using this, we are able to determine the total number of collectable sets that exist in the game, as well as find the maximum number of cards that can be played without having a collectable set. Furthermore, we can simulate the game using the Monte Carlo method to find the probability of having a collectable set in a random selection of cards from the deck.

1. The Game Set®

The game Set® was invented in 1974 by geneticist Marsha Jean Falco in an attempt to determine if epilepsy was an inherited disease in German Shepherds. She found through recording information about each dog that certain characteristics were grouped together. Instead of rewriting data, she decided to draw symbols on file cards to represent different gene combinations found in each dog. Her work of observing the different symbol combinations inspired her to create the game Set® [1].

Today, the game Set® is composed of 81 cards, each with a unique image. To play, one must lay down a 3x4 grid cards, face up. The purpose of the game is to collect three cards that meet a specific criteria, thus forming what we call a *collectable set*. Each card in the game has one quality from each of the four

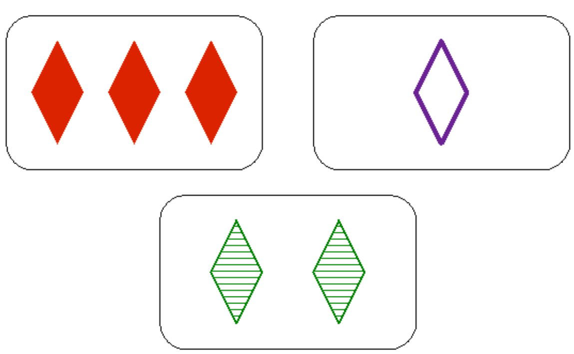


Figure 1. A *collectable set* containing all the same shape, and all different colors, number of symbols, and fills

categories, which are: color (green, purple, red), shape (oval, squiggle, diamond), fill (no fill, medium fill, solid), and the number of objects on the card (one, two, three). For instance, one card in the game has two solid green squiggles on it. To create a collectable set, one must find a collection of three cards in the grid for which all properties are the same, or each card has a different property, for each respective category. So a collectable set might have three cards that are all the same color, but each has a different shape, number, and fill. Clearly, each card cannot be the same for each of the four categories because there is only one card of that kind. After removing a collectable set, three more cards are placed down and the game continues until the deck is exhausted.

Figure 1 provides an example of a collectable set in the game. All three cards have the same shape (diamonds), but have all different characteristics for color, number, and fill.

2. Mathematical Introduction

While the game Set® is a source of entertainment for many, it is also rich in mathematical material. By relating the game to finite fields we can begin to answer questions which are provoked by the game.

In this article we will assume familiarity with linear algebra over any finite field. This means that we can talk about linear independence, dependence, vector spaces, subsets, and bases over a finite field just like we would over the real numbers.

We can create a one-to-one and onto correspondence between the vector space $\mathbb{F}_3^4$ and Set® by letting each card be represented by a point in $\mathbb{F}_3^4$. While there are multiple ways to construct this correspondence, we will be using the following notation for each card: $(color, number, shape, fill)$, where each entry is an element of $\mathbb{Z}_3$. We will let $0, 1$, and 2 represent the following for each card:

Color:	Number:	Shape:	Fill:
$0 =$Red	$0 = 1$ symbol	$0 =$Oval	$0 =$No fill
$1 =$Purple	$1 = 2$ symbols	$1 =$Diamond	$1 =$Medium fill
$2 =$Green	$2 = 3$ symbols	$2 =$Squiggle	$2 =$Solid

Each card is now a point in $\mathbb{F}_3^4$. For example the card with one unfilled red oval is given by the point $(0, 0, 0, 0)$. And since there are $3^4 = 81$ different entry combinations for a point in the field, and there are 81 unique cards in the deck, every point corresponds to a unique card. While each card is represented as a point, we also find that each collectable set in the game is actually a line in $\mathbb{F}_3^4$. Consider three cards which form a collectable set. For each category each trait is either all the same, that is the entries are all 0's, 1's, or 2's, or all traits are different, there is one entry of each 0, 1, and 2. Now taking the coordinates of each of the three points in a collectable set we find that when adding them (as elements of $\mathbb{Z}_3$), they all equal zero. This is true for each of the four entries of each point. Thus the cards represented by the points $x, y, z \in \mathbb{F}_3^4$ will satisfy the equation $x + y + z = 0$ if and only if the three cards are part of a collectable set. Later we will address that these three points are in fact on a line in $\mathbb{F}_3^4$.

In addition to observing the game, we will look at the creation of *magic squares.* A magic square is the collection of nine cards where each card is part of four collectable sets in the square. If we consider our vector space, but only over two dimensions, we are able to relate any magic square to $\mathbb{F}_3^2$, which is an affine plane of order three. Using the properties of affine planes, we can then determine what is necessary to create multiple magic squares using the remainder of the cards in the deck.

By using this correspondence between the cards in the game and the vector space over $\mathbb{F}_3$ we will be able to address the following questions regarding the game:

(1) Given any two cards in the game, are there multiple cards which can complete each collectable set?
(2) How many collectable sets exist in the game?
(3) Given a magic square using cards from the game, what is required to create a second magic square that has only one card in common with the first?
(4) What is the maximum number of cards which can be laid down without having a collectable set?

In the third section we will address the mathematical framework for the game. This will be an overview of projective geometry and some of the properties of affine planes. As stated earlier Set® is isomorphic to the vector space $\mathbb{F}_3^4$ and is in fact a four dimensional affine space of order three. Using properties addressed in this section we will have a better understanding of the proofs in subsequent sections.

The rest of the paper will be devoted to finding the answers to our questions involving the game. In the fourth section we will prove that for every pair of two cards drawn from the deck, there is a unique third card that completes the collectable set. In the fifth section we will count the total number of collectable sets contained within the game Set® using the relationship between $\mathbb{F}_3^4$ and the game. In the sixth section we will continue to discuss the properties of magic squares, and when given one magic square, what is required to create a second that intersects the first at one point.

The following section will address finding a *maximal d-cap*, which is the greatest number of points that can exist in $\mathbb{F}_3^d$ without creating any lines. We will apply this to the cases of $d = 2, 3, 4$, where the final proof will show that the maximal 4-cap, which is also the greatest number of cards we can have in the game without having a collectable set, is in fact twenty.

The eighth and final section will discuss outcomes of the game using the Monte Carlo method. We will address the probabilities of not having a collectable set given twenty cards from the game, as well as the average number of collectable sets that exist in a random set of twelve cards.

3. The Mathematical Framework

Before addressing the questions related to Set®, we must translate the game into a mathematical framework. In order to do so the topics of projective and affine geometry will be discussed.

First, consider a finite field of prime order q, denoted $\mathbb{F}_q$. Just like an n-dimensional vector space over the real numbers, there exists an n-dimensional vector space, $\mathbb{F}_q^n$ over this field. The vector spaces over a given field allow us to quantify the number of subspaces of a given dimension. Given $\mathbb{F}_q^n$, it is possible to determine the total number of vectors in the vector space.

Proposition 1. *The number of vectors in $\mathbb{F}_q^n$ is equal to q^n* [2].

Proof. In $\mathbb{F}_q$ there are q elements. Now suppose $\mathbb{F}_q$ is expanded to n-dimensions. Consider a vector in $\mathbb{F}_q^n$. The *ith* entry $(i = 1, 2, \ldots, n)$ of the vector has q possible choices. Since there are q possibilities in each of the n entries of the vector, there are q^n different combinations and thus the number of vectors in $\mathbb{F}_q^n$ is q^n.

□

We now consider the following important idea.

Definition 1. *A projective plane of order q consists of a set X of q^2+q+1 elements called points, and a set β of $(q+1)$-element subsets of β called lines, having the property that any two points lie on a unique line* [2].

It is not necessarily clear whether any number q will allow for the construction of a projective plane. The following proposition provides a method for exhibiting a projective plane of order q, when q is prime.

Proposition 2. *If q is a prime, then there is a projective plane of order q.*

Proof. Consider the field $\mathbb{F}_q$ containing q elements, where q is a prime number. We will define $V = \mathbb{F}_q^3$ to be the three dimensional vector space over the field. Let X be the set of one dimensional subspaces of V. Since there are $q^3 - 1$ non-zero vectors in V and there are $q - 1$ nonzero vectors which span any line in X, then there are

$$\frac{q^3 - 1}{q - 1} = q^2 + q + 1$$

elements in the set X, which we call points.

Now let T be a two-dimensional subspace of V, and we define L_T to be the following,

$$L_T = \{x \in X | x \subseteq T\}.$$

And we let $\beta = \{L_T | T \text{ is a 2-dimensional subspace of } V\}$. Since T is a two-dimensional subspace of V, it contains q^2 elements, and each nonzero vector of T spans the set. Since there are $q^2 - 1$ nonzero vectors in T and $q - 1$ of these vectors in L_T, then it follows that L_T has

$$\frac{q^2 - 1}{q - 1} = q + 1$$

elements.

For the last part, consider the points $x, y \in X$ where $x \neq y$. Then for $T = x + y$, we have a two-dimensional subspace of V such that L_T is the unique elements of β containing x and y.

□

A simple example of a projective plane of order one is a triangle consisting of three points. Since it has three points, we have $q^2 + q + 1 = 3$, and then we find $q = 1$. There are $q + 1 = 2$ points in every line. A triangle is a projective plane with an order of 1, and in fact, it is the only such plane. The following provides further properties of a projective plane of order q (see [2]).

Proposition 3. *In a projective plane of order q, the following hold:*

- *any point lies on $q + 1$ lines*
- *two lines meet in a unique point*
- *there are $q^2 + q + 1$ lines.*

Proof. Let p be a point in a projective plane of order q. Since there are q^2+q+1 points total, there are $q(q+1)$ points other than p. By Definition 1, every line through p will contain q additional points. Dividing by q, we get that there are $q+1$ lines through p. Now consider two lines, L_1 and L_2. Suppose that these two lines do not meet. Let p be a point on L_1. Then there are $q+1$ points on L_2, and by Definition 1 each must connect to p by a line other than L_1. But then there are $q+2$ lines through p, which is a contradiction, so L_1 must meet L_2. Let $|\beta|$ be the number of lines in the projective plane. Since there are $q+1$ points per line, we have that the number of points in X is

$$|\beta| \cdot (q+1) = (q^2+q+1)(q+1).$$

So then $|\beta| = q^2+q+1$. Therefore all three properties hold.

□

Here a representation of the projective plane of order 3:

0	1	2	3	4	5	6	7	8	9	T	J	Q
1	2	3	4	5	6	7	8	9	T	J	Q	0
5	6	7	8	9	T	J	Q	0	1	2	3	4
J	Q	0	1	2	3	4	5	6	7	8	9	T

Each column is a line in the projective space. All of the properties of a projective plane hold. There are thirteen points, thirteen lines, every point is contained on four lines, and there are four points on each line. Also note that each line has one, and only one point in common with each line, and therefore they all intersect.

A structure closely related to the projective plane is the *affine plane*. The following provides the definition of an affine plane of order q.

Definition 2. *An affine plane of order q consists of a set X of q^2 points, and a set β of q-element subsets of X called lines, such that two points lie on a unique line* [2].

Similarly to projective planes of order q, the following gives a method for constructing an affine plane.

Proposition 4. *If q is a prime, then there is a affine plane of order q.*

Proof. Let q be a prime number, and let $X = \mathbb{F}_q^2$. Since X is a two dimensional field of order q, it has q^2 elements. Suppose $v, z \in X$, and let $\beta = \{v + \alpha z : \alpha \in \mathbb{F}_q\}$. Since there are q different values for α, there are q different elements along each line in β. Now consider $z = w - v$, where $w \in X$. If $\alpha = 0$, we have $v + \alpha z = v + 0z = v$, and if $\alpha = 1$, we have $v + \alpha z = v + z = v + (w - v) = w$. We see that that the value $z = v - w$ determines the unique line that contains v and w. And following from Definition 2, we see that X is an affine plane of order q.

□

An example of an affine plane of order three is given in the following 3x3 grid of integers.

0	1	2
3	4	5
6	7	8

Here each row, column, diagonal, and off center diagonal represents a line in the affine plane. Writing out all of the lines (the vertical, horizontal, and diagonals), it becomes

0	1	2	0	3	6	0	1	2	2	1	0
3	4	5	1	4	7	5	3	4	3	5	4
6	7	8	2	5	8	7	8	6	7	6	8

where each column is a line. In regards to the definition, this is clearly an affine plane in which $q = 3$. There are $3^2 = 9$ points, 3 points make up a line, and each pair of numbers lies on a unique line. The following provides further properties of an affine plane.

Proposition 5. *In an affine plane of order q,*

- *any point lies on $q + 1$ lines;*
- *there are q(q+1) lines altogether*
- *(Euclids parallel postulate) if p is a point and L is a line, there is a unique line L' through p parallel to L*
- *parallelism is an equivalence relation; each parallel class contains q lines which partition the point set*

[2].

A proof of this proposition will not be provided here, but can be found in Cameron's text. Returning to the affine plane of order three, note that each point lies on $3 + 1 = 4$ lines, and there are $3 \cdot 4 = 12$ lines in the plane. There are also four parallel classes each containing three lines: the rows, the columns, the left to right diagonals, and the right to left diagonals.

Now that the definition and an example of an affine plane have been provided, the connection between projective space and affine space will be further explored. Let's return to the example of the projective plane of order three. By removing the line $\{0, 1, 5, J\}$, an affine plane of order three can be created.

•	•	2	3	4	•	6	7	8	9	T	•	Q
•	2	3	4	•	6	7	8	9	T	•	Q	•
•	6	7	8	9	T	•	Q	•	•	2	3	4
•	Q	•	•	2	3	4	•	6	7	8	9	T

After removing the first line, there is a plane consisting of nine points, twelve lines, and three points on each line. This is consistent with the definition of an affine plane where $q = 3$. By removing a line in the projective plane, an affine plane has been created. Each line in the projective plane was represented as a 4-tuple. By choosing the line $\{0, 1, 5, J\}$ to be the hyperplane which is removed, the original 4-tuples in the projective plane now have a unique representation as 3-tuples in the newly created affine space.

4. Cards in a Collectable Set

Given any two cards in the game Set®, is there only one card that can be added to complete a collectable set? When playing the game, this seems to be the case. If one picks two random cards, one can determine which are the four qualities in the third card that correspond to a collectable set. But is this always the case? By using the properties of $\mathbb{F}_3^4$, we are able to show this is true.

First we will define the term k-flat, which will come up frequently later on in this paper. Let V be a vector space over $\mathbb{F}_3^4$, then a k-flat is a k-dimensional subset of V. We will see that a collectable set corresponds to a 1-flat of the vector space $\mathbb{F}_3^4$, and in the sixth section we will see that a magic square is in fact a 2-flat in $\mathbb{F}_3^4$.

The following lemma will help us to show that each collectable set is a 1-flat in $\mathbb{F}_3^4$.

Lemma 1. *Three points $x, y, z \in \mathbb{F}_3^4$ are on a line if and only if $x + y + z = 0$.*

Proof. Let $x, y, z \in \mathbb{F}_3^4$ such that x, y, z are on a line. Because they are on a line we can rewrite each in the form $\upsilon_o + \alpha\omega_o$, where $\alpha = 0, 1, 2$. Then let

$$x = \upsilon_o$$
$$y = \upsilon_o + \omega_o$$
$$z = \upsilon_o + 2\omega_o.$$

Adding the three points we find $x + y + z = 3\upsilon_o + 3\omega_o = 0$.

Now suppose $x, y, z \in \mathbb{F}_3^4$ such that $x + y + z = 0$. Let $x = \upsilon_o$ and $y = x + \omega_o$. This implies that $y = \upsilon_o + \omega_o$ and $\upsilon_o + 2\omega_o = x + 2(y - x) = -x + 2y = -x - y = z$, and we have

$$x = \upsilon_o$$
$$y = \upsilon_o + \omega_o$$
$$z = \upsilon_o + 2\omega_o.$$

We conclude that x, y, z are on a line. This completes the proof.

□

Theorem 1. *For any two cards drawn from the deck, there exists a unique third card that, with the other two, creates a collectable set.*

Proof. Let the cards in the game be points in $\mathbb{F}_3^4$. Let $x, y \in \mathbb{F}_3^4$. To show that there exists a point which completes a line with x and y, consider the point $z = -x - y$. By using the properties of vector spaces, we find

$$x + y + z = x + y - x - y = 0$$

which, by Lemma 1, implies that x, y, z are all on a line. Thus given any two points in $\mathbb{F}_3^4$, there exists a third point which completes the line. To prove the uniqueness of the third card, we will use contradiction. Assume that there exist two distinct points $z_1, z_2 \in \mathbb{F}_3^4$ which both complete a line with x and y. Then we have

$$x + y + z_1 = 0$$
$$x + y + z_2 = 0$$

which implies

$$z_1 = 0 - x - y$$
$$z_2 = 0 - x - y$$

and it follows that $z_1 = z_2$. Therefore there is only one point in $\mathbb{F}_3^4$ that completes a line with the points x and y. Consequently, for any two cards in the game, there exists a unique card that completes the collectable set.

□

5. Cards and Lines

To determine the total number of collectable sets of cards in the game, we will first consider the total number of lines in $\mathbb{F}_3^2$, $\mathbb{F}_3^3$, and then look at $\mathbb{F}_3^4$.

5.1. **Lines in $\mathbb{F}_3^2$.** The vector space $\mathbb{F}_3^2$ is an affine plane of order three. From Definition 2 we know there are nine points, and any two points lie on a unique line. Then there are

$$\binom{9}{2}$$

different lines. However, this value over counts the number of lines (or rather the number of triples) by a factor of three, thus it must be divided by three. For example, given the collectable set (A, B, C), we can pick the three pairs $\{A, B\}, \{A, C\}, \{B, C\}$, which each determine the same line. Dividing by three, this gives

$$\binom{9}{2}/3 = 12$$

as the total number of lines in $\mathbb{F}_3^2$, which is equivalent to the total number of collectable sets in any magic square.

5.2. **Lines in $\mathbb{F}_3^3$.** Similarly we can determine the total number of lines in the vector space $\mathbb{F}_3^3$. By Proposition 1, there are a total of 27 points. Using the same method as above, we find that there are

$$\binom{27}{2}/3 = 117$$

lines in $\mathbb{F}_3^3$.

5.3. **Lines in $\mathbb{F}_3^4$.** Finally to determine the total number of collectable sets in the game we consider the number of lines in $\mathbb{F}_3^4$. Using Proposition 1, there are $3^4 = 81$ points in $\mathbb{F}_3^4$ and there are $\binom{81}{2}$ total ways to choose two points in the field. To adjust for over counting, we divide by three and find that there are

$$\binom{81}{2}/3 = 1080$$

lines in $\mathbb{F}_3^4$, and thus a total of 1080 collectable sets in the deck of 81 cards.

6. Magic Squares

Within the game of Set®, we can construct what is called a *magic square*. A magic square is formed by taking three cards, which we will denote by $x, y, z \in \mathbf{F}_3^4$, and placing them according to the grid below.

	y	
x		z

We will call this grid P. Using the rules of Set®, we can find the unique cards that complete each collectable set with the pairs (x, y), (x, z), and (y, z). As mentioned earlier, the cards required to complete each collectable set are those that, when added to the other two, equal 0 (mod 3). To complete the 3x3 grid of cards, we fill the spaces in accordingly,

$-(y+z)$	$-y+x+z$	$-(x+y)$
$-x+y+z$	y	$x+y-z$
x	$-(x+z)$	z

so that each row, column, and diagonal create a collectable set. We note that the cards which complete P are dependent on the original three x, y, and z.

While the construction of each magic square seems to work with random cards, we must actually avoid certain cards. For example, as stated in Section 2 the point $(0, 0, 0, 0)$ represents the single red unfilled oval. Then we can create the following magic square,

$S_1 =$

$(2,2,0,0)$	$(1,2,0,0)$	$(0,2,0,0)$
$(1,1,0,0)$	$(\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{0})$	$(2,1,0,0)$
$(0,0,0,0)$	$(2,0,0,0)$	$(\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{0})$

where $(0, 1, 0, 0)$ and $(1, 0, 0, 0)$ are the two cards that were initially set down to complete the square. But then can any two cards be placed on the grid? The answer is no. For example, if the cards $(1, 0, 0, 0)$ and $(2, 0, 0, 0)$ are set down on the grid, we have

$(0,0,0,0)$	$(2,0,0,0)$	$(1,0,0,0)$
$(0,0,0,0)$	$(\mathbf{2},\mathbf{0},\mathbf{0},\mathbf{0})$	$(1,0,0,0)$
$(0,0,0,0)$	$(2,0,0,0)$	$(\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{0})$

which violates the rules of the game; specifically, there are no duplicates. We note that in $\mathbf{F}_3^4$, the points $(1, 0, 0, 0)$ and $(2, 0, 0, 0)$ are linearly dependent, while $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$ are independent. If we choose any two linearly dependent cards, that is cards represented by linearly dependent, or equivalently, collinear vectors in $\mathbf{F}_3^4$, we will not be able to create a magic square. To prove this, consider the square with 0, x, and $2x$. Because we are in $\mathbf{F}_3^4$, the following will occur

$-(2x+x)$	$2x-x$	$-(x+0)$
$2x+x$	x	$-(2x+x+2x)$
0	$-2x$	$2x$

Which is equivalent to

0	x	$2x$
0	x	$2x$
0	x	$2x$

So we can't create a magic square with collinear initial vectors.

6.1. **Set® as an Affine Plane.** The following will allow us to see Set® is an affine plane, and that we can partition the cards into ten magic squares that intersect at one point. First, consider an abelian group G, of order n^2, which has $n+1$ subgroups, $H_0, \cdots, H_n$ of order n, such that the intersection of the subgroups is $\{0\}$. Then, if we let the elements of G be points, and the cosets of H_i for all $i = 0, 1, \cdots, n$ be our lines, then we have an affine plane. To show that two points are contained in a line, consider $x, y \in G$. Consider the subgroup H_k that contains $x - y$. We know that both x and y must be in the coset $H_k + y$. So for the two points, x and y there is a line $H_k + y$, and in fact unique line, that contains them. If we continue with some more properties of affine planes, we note that there are $n+1$ subgroups, and thus $n+1$ lines. If we remove $\{0\}$, we see that each subgroup contains $n^2 - 1/(n+1) = n - 1$ elements, and therefore there are n elements including $\{0\}$ in each subgroup.

Next can show that Set® is an affine plane of order nine. There are 9^2 cards that can be represented as points in $\mathbf{F}_3^4$. We can deconstruct the cards into 10 subgroups such that the only card they share is $(0,0,0,0)$, that is the single red unfilled oval. Here we write the subgroups as given in [3],

$$
\begin{aligned}
H_0 &= \{(0000), (1000), (2000), (0100), (0200), (1100), (2200), (2100), (1200)\}\\
H_1 &= \{(0000), (0010), (0020), (0001), (0002), (0011), (0022), (0021), (0012)\}\\
H_2 &= \{(0000), (1010), (2020), (0101), (0202), (1111), (2222), (2121), (1212)\}\\
H_3 &= \{(0000), (2010), (1020), (0201), (0102), (2211), (1122), (1221), (2112)\}\\
H_4 &= \{(0000), (0110), (0220), (2001), (1002), (2111), (1222), (2221), (1112)\}\\
H_5 &= \{(0000), (0210), (0120), (1001), (2002), (1211), (2122), (1121).(2212)\}\\
H_6 &= \{(0000), (1110), (2220), (2101), (1202), (0211), (0122), (1021), (2012)\}\\
H_7 &= \{(0000), (2210), (1120), (1201), (2102), (0111), (0222), (2021), (1012)\}\\
H_8 &= \{(0000), (2110), (1220), (2201), (1102), (1011), (2022), (0121), (0212)\}\\
H_9 &= \{(0000), (1210), (2120), (1101), (2202), (2011), (1022), (0221), (0112)\}
\end{aligned}
$$

Each subgroup H_i is closed under addition modular 3. If we go back to our magic square, we see that each of these subgroups represents an individual magic square. Here is a possible way to have ten magic squares that intersect at one card.

6.2. **Creating Multiple Magic Squares.** Now suppose that we have one magic square, S_1. What are the conditions that must be met in order to create a second which intersects the first at a single value? That is given one magic square, S_1, containing the element x, can we find a second magic square S_2 such that $S_1 \cap S_2 = \{x\}$? Since our two vectors in S_1 are linearly independent, S_1 is 2-dimensional, and it is also a subspace of $\mathbf{F}_3^4$, we can create the transformation[1] $\phi : \mathbf{F}_3^4 \to \mathbf{F}_3^4$, defined

[1]Making the magic square into a subspace, and the creation of this transformation was contributed by Professor Patrick Keef

as the following,

$$
\begin{aligned}
\phi(1,0,0,0) &= (1,2,1,2)\\
\phi(0,1,0,0) &= (0,1,1,1)\\
\phi(0,0,1,0) &= (0,0,1,0)\\
\phi(0,0,0,1) &= (0,0,0,1)
\end{aligned}
$$

Because our basis vectors map to a new set of linearly independent vectors, we will have a magic square. Our transformation matrix is the following,

$$
\begin{vmatrix}
1 & 0 & 0 & 0\\
2 & 1 & 0 & 0\\
1 & 1 & 1 & 0\\
2 & 1 & 0 & 1
\end{vmatrix}
$$

and by multiplying it with each element of S_1 we can create a second square, S_2.

$$
S_2 = \begin{array}{c|c|c}
(2,0,1,0) & (1,1,0,1) & (0,2,2,2)\\ \hline
(1,0,2,0) & (\mathbf{0},\mathbf{1},\mathbf{1},\mathbf{1}) & (2,2,0,2)\\ \hline
(0,0,0,0) & (2,1,2,1) & (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})
\end{array}
$$

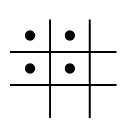
Thus we can construct a second magic square from one magic square. From the previous subsection, we discovered that the elements of $\mathbb{F}_3^4$ can be partitioned into ten subgroups $H_0, H_1, \cdots, H_9$, such that there exists one element x such that for $i \neq j$, $H_i \cap H_j = \{x\}$. Since these subgroups correspond to magic squares in the game, we see that we can in fact partition the cards into ten magic squares that have only one card in common.

7. Maximal d-caps

In "The Card Game Set" the authors define the term *d-cap* to be a subset of $\mathbf{F}_3^d$ which does not contain any lines. In terms of the card game Set®, it is the maximum number of cards that can be drawn without having a collectable set. In the previous section we considered creating magic squares, which are in $\mathbf{F}_3^2$. Consider the single red cards, then the maximal 2-cap will be the largest set of single red cards that can be placed in a 3x3 grid without having a collectable set; or in $\mathbf{F}_3^2$ it is the maximum number of points that can exist without forming a line. Here we establish the value of the maximal 2-cap.

Proposition 6. *There are four points in a maximal* 2*-cap.*

Proof. To prove that the 2-cap is four, we must first show that there exists a 2-cap which contains four points, and that anymore than four points will create a line. We can prove the first by example. The following grid has four points, and no lines.



Therefore there are at least four points without having a line. Now we must show that for any set of five points, we will have a line. By contradiction, assume that we can place five points onto the grid without creating a line. First, because we cannot have three points next to each other, one row can only contain one point, and the others can contain two. Without loss of generality, assume the first row contains only 1 point, where the • represents our point, and 0 represents the absence of points.

•	0	0

Here we note that each point is on four lines. Because the first row contains no other points, there are three lines through our first point on which to place our other four points. This leads to a pigeon hole argument. The first three points can each be placed on a different line. Then we have two points on each line. But then the fifth point can only go on one of these three lines, and therefore we will have a line containing three points, and therefore a contradiction. Thus, there are four points in a maximal 2-cap.

□

Before proving the values of the 3-cap and the 4-cap we will provide an important result which allows us to count the number of hyperplanes containing a k-flat. The term *hyperplane* is used frequently in the following propositions. If we let V be an n-dimensional vector space, then a hyperplane H is an $(n-1)$-flat of V. Next we define the term *N-marked plane*. Let H be a hyperplane, and C be a d-cap in $\mathbb{F}_3^d$, then an *N-marked* plane is the ordered pair (H, M) where M is an N-element subset of $H \cap C$. We will only see the cases of *2*-marked and *3*-marked planes in this section.

Proposition 7. [4] *The number of hyperplanes containing a fixed k-flat in* $\mathbb{F}_3^d$ *is given by*

$$\frac{3^{d-k}-1}{2}.$$

Proof. Let K be a k-flat that contains the origin. We can create the following bijection,

$$\mathbb{F}_3^d \to \mathbb{F}_3^d/K \cong \mathbb{F}_3^{d-k}$$

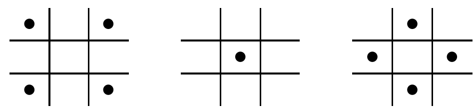
where we remove k dimensions, and are left with a $(d-k)$-dimensional vector space. This maps a hyperplane containing K to a hyperplane in $\mathbb{F}_3^{d-k}$, which contains the origin.

Because the mapping is a bijection, the number of hyperplanes in $\mathbb{F}_3^d$ that contain K is equivalent to the number of hyperplanes in $\mathbb{F}_3^{d-k}$ that contain the origin. Since each plane is determined by a nonzero normal vector and there are two nonzero normal vectors for every hyperplane, there are half as many hyperplanes as nonzero vectors in $\mathbb{F}_3^{d-k}$. It follows that the number of hyperplanes containing the origin is

$$\frac{3^{d-k}-1}{2}.$$

□

Next, we can show that a maximal 3-cap contains nine points. First we provide an example of a 3-cap containing nine points.



Proposition 8. *A maximal* 3*-cap has nine points* [4].

Proof. We have seen an example of a 3-cap containing nine points. Now we must show that any set of more than nine points contains a line in $\mathbb{F}_3^3$. By contradiction, assume that there does exist a 3-cap containing ten points. The space $\mathbb{F}_3^3$ can be divided into three parallel planes, which, by Proposition 6, contain at most four points each. If we have a set C of ten points in $\mathbb{F}_3^3$, then the plane with the least number of points has either two or three points. Let H be the plane with the least number of points, and let a and b be the two points in H. Here we see H is a *2*-marked plane. Then there are seven other points, $x_1, x_2, x_3, x_4, x_5, x_6$, and x_7 in $\mathbb{F}_3^3$ which are not in H. Since a and b are in H, by Proposition 7 there are 3 other planes, let's call them M_1, M_2, and M_3 which contain both a and b as well. Then the points $x_1, x_2, x_3, x_4, x_5, x_6$, and x_7 lie in the three planes, and by the pigeon hole principle, three of these points must lie in one M_i. But M_i contains a and b as well and thus has 5 points, which contradicts Proposition 6. Therefore there are at most nine points in a 3-cap.

□

In order to prove the value of the maximal 4-cap, we need a stronger method, which we will use to again prove that a maximal 3-cap has nine points.

Proposition 9. *A maximal* 3*-cap has nine points* [4].

Proof. From our previous example, we know there are at least nine points in a maximal 3-cap. Suppose there exists a 3-cap with ten points. Let C be our 3-cap. Then let $\mathbb{F}_3^3$ be the union of three parallel planes, H_1, H_2, and H_3. Then we can create an unordered hyperplane triple $\{|C \cap H_1|, |C \cap H_2|, |C \cap H_3|\}$, where each $|C \cap H_i|$ is the number of points in each plane contained in C. Since each plane has a maximal 2-cap of four, the possible triples are of the form $\{4, 4, 2\}$ and $\{4, 3, 3\}$. Let a and b be as follows:

$$a = \text{the number of hyperplane triples of the form } \{4,4,2\}$$
$$b = \text{the number of hyperplane triples of the form } \{4,3,3\}$$

Where a and b must both be non-negative. Then there are $a + b$ different ways to decompose $\mathbb{F}_3^3$ as the union of three hyperplanes. To find $a + b$ we can consider the number of lines through the origin. For each family of three parallel hyperplanes there is a unique line through the origin and perpendicular to the hyperplanes. Here we note the origin is a 0-flat, and by Proposition 7 when $k = 0$ there are thirteen hyperplane triples which contain the origin. Thus there are thirteen ways to decompose the union of three hyperplanes, and we get $a + b = 13$.

Next we look at the *2*-marked planes, that is planes with at least two points which are contained in C. Since there are four planes that contain any pair of points (as in the proof of Proposition 8) and ten choose two different pairs of points, there are $4\binom{10}{2} = 180$ *2*-marked planes.

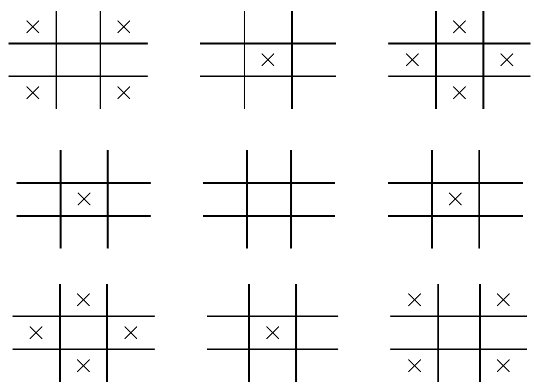
For each hyperplane triple of the form $\{4,4,2\}$, there are $\binom{4}{2} + \binom{4}{2} + \binom{2}{2} = 13$ *2*-marked planes. For each hyperplane triple of the form $\{4,3,3\}$ there are $\binom{4}{2} + \binom{3}{2} + \binom{3}{2} = 12$ *2*-marked planes. So then we have

$$13a + 12b = 180.$$

Using $a + b = 13$, we find that $b = -11$ and $a = 24$. But this is a contradiction because a and b cannot be negative. Therefore there are at most nine points in a maximal 3-cap.

□

Now we turn to the computation of a maximal 4-cap. Here is an example of a 4-cap containing 20 points [4]:



Here we see that there are no lines. However, if we add another point anywhere in the grid, we will have a line. This leads us to the following proposition.

Proposition 10. *A maximal* 4*-cap has twenty points* [4].

Proof. By contradiction, assume we have a 4-cap C_4 of twenty-one points. Now let x_{ijk} be the number of three-dimensional hyperplane triples of the form $\{i, j, k\}$, where i, j, k represent the number of points in the corresponding parallel hyperplanes (note $i + j + k = 21$). Since the maximal 3-cap is nine points, there are seven different hyperplane triples:

$$\{9,9,3\}, \{9,8,4\}, \{9,7,5\}, \{9,6,6\}, \{8,8,5\}, \{8,7,6\}, \{7,7,7\}$$

By Proposition 7, the number of groups of three parallel hyperplanes in $\mathbb{F}_3^4$ is equal to the number of lines that go through the origin, given by $(3^4 - 1)/2 = 40$. It follows that the sum of our different hyperplane forms is forty, and we have,

$$x_{993} + x_{984} + x_{975} + x_{966} + x_{885} + x_{876} + x_{777} = 40. \quad (1)$$

Next we consider the *2*-marked planes, the planes where at least two points are contained in C_4. Since two points are contained in a 1-flat, by Proposition 7, there are $(3^3 - 1)/2 = 13$ *2*-marked planes. The following argument is similar to Proposition 9. Since there are thirteen hyperplanes with distinct pairs of points, there are $13\binom{21}{2} = 2730$ *2*-marked planes. For a hyperplane triple of the form $\{i, j, k\}$ there are $\binom{i}{2} + \binom{j}{2} + \binom{k}{2}$ *2*-marked planes. We can formulate a second equation based off of the these values.

$$\left[\binom{9}{2}+\binom{9}{2}+\binom{3}{2}\right]x_{993}+\left[\binom{9}{2}+\binom{8}{2}+\binom{4}{2}\right]x_{984}+\left[\binom{9}{2}+\binom{7}{2}+\binom{5}{2}\right]x_{975}+$$
$$\left[\binom{9}{2}+\binom{6}{2}+\binom{6}{2}\right]x_{966}+\left[\binom{8}{2}+\binom{8}{2}+\binom{5}{2}\right]x_{885}+\left[\binom{8}{2}+\binom{7}{2}+\binom{6}{2}\right]x_{876}+$$
$$\left[\binom{7}{2}+\binom{7}{2}+\binom{7}{2}\right]x_{777}=2730.$$

This simplifies to

$$(2)\qquad 75x_{993}+70x_{984}+67x_{975}+66x_{966}+66x_{885}+64x_{876}+63x_{777}=2730.$$

Next, we consider the *3*-marked planes, which are the hyperplanes containing three distinct points of C_4. Since three noncollinear points make up a 2-flat, by Proposition 7, there are $(3^2-1)/2=4$ hyperplanes with three distinct points from the cap. Since there are twenty-one choose three different ways to select three distinct points, there are $4\binom{21}{3}=5320$ *3*-marked planes. This gives the equation,

$$\left[\binom{9}{3}+\binom{9}{3}+\binom{3}{3}\right]x_{993}+\left[\binom{9}{3}+\binom{8}{3}+\binom{4}{3}\right]x_{984}+\left[\binom{9}{3}+\binom{7}{3}+\binom{5}{3}\right]x_{975}+$$
$$\left[\binom{9}{3}+\binom{6}{3}+\binom{6}{3}\right]x_{966}+\left[\binom{8}{3}+\binom{8}{3}+\binom{5}{3}\right]x_{885}+\left[\binom{8}{3}+\binom{7}{3}+\binom{6}{3}\right]x_{876}+$$
$$\left[\binom{7}{3}+\binom{7}{3}+\binom{7}{3}\right]x_{777}=5320$$

which simplifies to

$$(3)\ 169x_{993}+144x_{984}+129x_{975}+124x_{966}+122x_{885}+111x_{876}+105x_{777}=5320.$$

Next we want to solve for our x_{ijk} values. By taking 693 times Equation (1) plus 3 times Equation (3), then subtracting 16 times Equation (2), we get

$$(693+507-1200)x_{993}+(693+432-1120)x_{984}+(693+387-1072)x_{975}+$$
$$(693+372-1056)x_{966}+(693+366-1056)x_{885}+(693+333-1024)x_{876}+$$
$$(693+315-1008)x_{777}=27720+15960-43680$$

which is equivalent to

$$5x_{984}+8x_{975}+9x_{966}+3x_{885}+2x_{876}=0.$$

Since the values of x_{ijk} must be non-negative, then $x_{984}=x_{975}=x_{966}=x_{885}=x_{876}=0$ is the only solution. Then if we also consider 2 times Equation (2) minus 63 times Equation (1), we get

$$12x_{993}+7x_{984}+4x_{975}+3x_{966}+3x_{885}+x_{876}=210.$$

Substituting our zero values, we get $x_{993}=35/2$, but then x_{993} cannot be an integer, which is a contradiction because we can only have integral values for the number of hyperplanes. Thus, we cannot have twenty-one points in a 4-cap, and therefore there are twenty points in a maximal 4-cap.

□

8. Monte Carlo Method

Another way to address some questions related to the game is to simulate Set® using the Monte Carlo method. The following table provides the average number, out of 10,000 trials, of collectable sets contained in a random set of drawn cards, as well as the probability that the set of cards will contain a collectable set.

Cards drawn	Probability of having a collectable set	Average number of collectable sets
3	0.0134	0.0134
4	0.0536	0.0536
5	0.1218	0.1238
6	0.2306	0.2472
7	0.3887	0.4525
8	0.5462	0.7128
9	0.7024	1.0559
10	0.8332	1.4945
11	0.9183	2.0915
12	0.9665	2.7958
15	0.9993	5.7485
18	1	10.3281
21	1	16.8553

Looking at this data in the table above, we see that although we are not guaranteed to have a collectable set until 21 cards are drawn, we get 100% chance of having a collectable set with 18 cards. Clearly, the simulation will not provide us with the exact percent of having a collectable set, but it can give us an estimate. Considering the game Set®, twelve cards are placed in the center, and about 96.65% of the time there will be a collectable set. So roughly 3.35% of the time the dealer will have to place an additional three cards down.

We can draw several other conclusions from these results. In order to have about 50% chance of getting a collectable set one needs to draw eight cards from the deck. It is not until nine cards are drawn that we have at least one collectable set on average. Although the game doesn't begin until there are twelve cards laying face up, the existence of a set impacts the dealer, for he or she does not have the ability to closely analyze the cards while continuing to deal. Again from Proposition 10 we know that there is at least one collectable set given any 21 cards from the deck. However, from this data we see that there are roughly 16 collectable sets on average.

Using probability we know that given three cards, there is a 1/79, or 0.0127, chance of having a collectable set. This is because for any two cards, there is only one other card out of 79 that will complete a collectable set. Now compare this to the simulated probability. The program calculated that there would be 0.0134 chance of having a set, which only differs by 0.0007.

9. Conclusion and Suggestions for Further Research

In this paper we have shown how we can use the isomorphism between the game Set® and the vector spaces over $\mathbb{F}_3$ to show that collectable sets correspond to lines

(1-flats) and magic squares correspond to planes (2-flats) in $\mathbb{F}_3^4$. This isomorphism also allows us to connect the game to the study of projective and affine planes. Furthermore, we are able to determine the maximal 2-cap, 3-cap, and 4-cap values. Relating this to the game, we are able to conclude that once 21 cards are drawn from the deck, the existence of a collectable set is guaranteed.

While we have strictly discussed $\mathbb{F}_3^d$, for the values $d = 2, 3, 4$, further exploration could be continued for values $d \geq 5$. In terms of the game, it would be as though the cards had at least one additional category. For instance, the cards could have a background color category, a texture category, or maybe even a scent category (though the last might be the least practical). A question we might ask is what is the maximal 5-cap? Is it possible to determine using the methods in the proof of Proposition 10? Using Fourier analysis, Davis and Maclagan show in their paper that the maximal 5-cap is in fact 45 [4].

Another area of interest is considering what happens when we change the order of our field. What if each category of the game had five possible traits instead of three? Since five is prime, we know there exists the field of order five. Then we could find a correspondence between the new cards and points in $\mathbb{F}_5^4$. Since each entry has five possible values, the game would be composed of 625 cards. Mathematically this would be an interesting idea to consider; however, in terms of the game, 625 might be too many cards with which to play.

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