1. Give an example showing that division with remainder need not be unique in a Euclidean domain.

2. Prove that in a principle ideal domain $R$, every pair $a, b$ of elements, not both zero, has a greatest common divisor $d$, with these properties:
   (i) $d = ar + bs$ for some $r, s \in R$;
   (ii) $d$ divides $a$ and $b$;
   (iii) if $e \in R$ divides $a$ and $b$, it also divides $d$.
Moreover, $d$ is determined up to unit factor.
(This problem basically asks for a detailed proof of Proposition 12.2.8 in Artin).

3. If $a, b$ are integers and $a$ divides $b$ in the ring of Gauss integers, then $a$ divides $b$ in $\mathbb{Z}$.

4. Prove that the factorizations $2 = (1 + i)(1 - i)$ and $5 = (2 + i)(2 - i)$ are prime factorizations of 2 and 5 in $\mathbb{Z}[i]$, respectively.

5. (Artin 12.3.2) Prove that two integer polynomials are relatively prime elements of $\mathbb{Q}[x]$ if and only if the ideal they generate in $\mathbb{Z}[x]$ contains a nonzero integer.

6. We showed already that $(2, x)$ is not a principal ideal in $\mathbb{Z}[x]$. This shows that $\mathbb{Z}[x]$ is not a PID.
   (i) Show that $(2, x)$ is principal in $\mathbb{Q}[x]$. Which element generates $(2, x)$ in $\mathbb{Q}[x]$?
   (ii) What is $(2, x)$ in $(\mathbb{Z}/p\mathbb{Z})[x]$ where $p$ is prime? For which $p$ is $(2, x)$ maximal?

7. (i) Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$. Suppose $r/s \in \mathbb{Q}$ is a root of $p(x)$ where $r$ and $s$ are coprime. Then $r | a_0$ and $s | a_n$.
   (ii) Use part (i) to show that $x^3 - 3x - 1$ is irreducible in $\mathbb{Z}[x]$. 