Homework 4
due May 2, 2014 in class

(1) (a) (Artin 14.1.2) Let V be an abelian group. Prove that if V has the structure of a $\mathbb{Q}$-module with its given law of composition as addition, then this structure is uniquely determined.
  (b) Prove that no finite abelian group has a $\mathbb{Q}$-module structure.

(2) The *annihilator* of an R-module V is the set $I = \{r \in R \mid rV = 0\}$.
  (a) Prove that I is an ideal of R.
  (b) What is the annihilator of the $\mathbb{Z}$-module $\mathbb{Z}/(2) \times \mathbb{Z}/(3) \times \mathbb{Z}/(4)$? What is the annihilator of the $\mathbb{Z}$-module $\mathbb{Z}$?

(3) (Artin 14.2.1) Let $R = \mathbb{C}[x, y]$, and let M be the ideal of R generated by the two elements (x, y). Prove or disprove: M is a free R-module.

(4) Let A be an $n \times n$ matrix with coefficients in a ring R, let $\varphi : R^n \to R^n$ be left multiplication by A, and let $d = \det A$. Prove or disprove: The image of φ is equal to dR^n.

(5) Let I be an ideal of a ring R. Prove that I is a free R-module if and only if it is a principal ideal, generated by an element α which is not a zero divisor in R.

(6) (Artin 14.4.1(c)) Determine integer matrices Q^{-1}, P which diagonalize the matrix

$$A = \begin{pmatrix} 4 & 7 & 2 \\ 2 & 4 & 6 \end{pmatrix}.$$

(7) Let $d_1, d_2, \ldots$ be the integers referred to in Theorem 14.4.6 in Artin.
  (a) (Artin 14.4.2) Prove that d_1 is the greatest common divisor of the entries a_{ij} of A.
  (b) Prove that $d_1 d_2$ is the greatest common divisor of the determinants of the 2×2 minors of A.