Homework 6
due May 23, 2014 in class

(1) (Artin 4.7.2) Prove that

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

is an idempotent matrix, and find its Jordan form.

(2) (Artin 4.7.3) Let V be a complex vector space of dimension 5, and let T be a linear operator on V which has characteristic polynomial $(t-\alpha)^5$. Suppose that the rank of the operator $T - \alpha I$ is 2. What are the possible Jordan forms for T?

(3) Show that every complex $n \times n$ matrix is similar to a matrix of the form $D + N$, where D is diagonal, N is nilpotent, and $DN = ND$.

(4) Prove the Cayley-Hamilton Theorem, that if $p(t)$ is the characteristic of an $n \times n$ matrix A, then $p(A) = 0$.

(5) Let F be a field containing exactly eight elements. Prove or disprove: The characteristic of F is 2.

(6) Let α be an algebraic element over a field F, and let $f(x)$ be its irreducible polynomial of degree n. Prove that $(1, \alpha, \alpha^2, \ldots, \alpha^{n-1})$ is a basis of $F[\alpha]$ as a vector space over F.

(7) Prove that $x^3 + x^2 + 1$ is irreducible in $\mathbb{Z}/2\mathbb{Z}[x]$. Use this polynomial to construct a field of order 8. What is the order of its multiplicative group? Describe the group explicitly.