Homework 7

due May 30, 2014 in class

(1) (Artin 15.2.1) Let α be a complex root of the irreducible polynomial $x^3 - 3x + 4$. Find the inverse of $\alpha^2 + \alpha + 1$ in the form $a + b\alpha + c\alpha^2$, $a, b, c \in \mathbb{Q}$.

(2) (Artin 15.3.1) Let F be a field, and let α be an element which generates a field extension of F of degree 5. Prove that α^2 generates the same extension.

(3) Let K be a field generated over F by two elements α, β of relatively prime degrees m, n, respectively. Prove that $[K : F] = mn$.

(4) (a) Let $F \subset F' \subset K$ be field extensions. Prove that if $[K : F] = [K : F']$, then $F = F'$.
(b) Give an example showing that this need not be the case if F is not contained in F'.

(5) Prove or disprove: Every algebraic extension is a finite extension.

(6) (Artin 15.6.1) Let F be a field of characteristic zero, let f' denote the derivative of a polynomial $f \in F[x]$, and let g be an irreducible polynomial which is a common divisor of f and f'. Prove that g^2 divides f.

(7) Let $f(x)$ be an irreducible polynomial of degree n over a field F. Let $g(x)$ be any polynomial in $F[x]$. Prove that every irreducible factor of the composite polynomial $f(g(x))$ has degree divisible by n.