Homework 3

due May 9

Problem 1. The Lie algebra $\mathfrak{sl}_2$ consists of traceless 2×2 matrices (over $\mathbb{R}$ or over $\mathbb{C}$) and has basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Compute the commutators $[H, E]$, $[H, F]$ and $[E, F]$.

Problem 2. The Lie algebra $\mathfrak{so}_3$ consists of real skew-symmetric 3×3 matrices. Find the basis in $\mathfrak{so}_3$ and compute all commutators in this basis.

Problem 3. The Lie algebra $\mathfrak{su}_2$ consists of complex 2×2 matrices X such that $X^\star = -X$ and $\mathrm{Tr}(X) = 0$. Find the basis in $\mathfrak{su}_2$ over $\mathbb{R}$ and compute all commutators in this basis.

Problem 4. Use Problems 1-3 to prove the following:

(1) The Lie algebras $\mathfrak{so}_3$ and $\mathfrak{su}_2$ are isomorphic.
(2) The complexified Lie algebras $\mathfrak{so}_3 \otimes \mathbb{C}$, $\mathfrak{su}_2 \otimes \mathbb{C}$ and $\mathfrak{sl}_2(\mathbb{R}) \otimes \mathbb{C} = \mathfrak{sl}_2(\mathbb{C})$ are all isomorphic.

Problem 5. Show that $U(n)$ and $SU(n) \times U(1)$ are not isomorphic as groups.
Hint: Compute the centers $Z(U(n))$ and $Z(SU(n) \times U(1))$.

Problem 6. Let X and Y be $n \times n$ matrices. Show by induction that

$$(\mathrm{ad}_X)^m(Y) = \sum_{k=0}^{m} \binom{m}{k} X^k Y (-X)^{m-k},$$

where

$$(\mathrm{ad}_X)^m(Y) := \underbrace{[X, \ldots [X, [X, Y]] \ldots]}_{m}.$$

Now show by direct computation that

$$e^{\mathrm{ad}_X}(Y) = \mathrm{Ad}_{e^X}(Y) := e^X Y e^{-X}.$$

Assume that it is legal to multiply power series term by term.